MA268 Second Year Essay

VARIATIONAL PRINCIPLES AND NOETHER'S THEOREM

Contents

1 Introduction and motivating example

The notion of extremising a quantity, as a function of multiple variables, is a well-established concept in mathematics. Many problems in Physics, Computing and Geometry involve maximising or minimising various quantities subject to constraints. The process of finding the roots of the derivative of some objective function is a familiar one, but what happens when the path of optimisation is no longer so straightforward?

- Given any traversable surface, what path should one take in order to minimise the distance travelled between two fixed points?
- What shape does a suspended chain naturally make in order to minimise its gravitational potential energy?
- What shape fence should a farmer build in order to maximise the grazing area for his sheep if he has a finite amount of fencing to use?

Questions like these require a more delicate approach to extremisation, so we must study new principles to tackle them. We will also explore the unique scenarios that arise when the quantity we want to extremise exhibits local 'symmetries', and the implications this has in fields such as mathematical physics. Pillars of mechanics like conservation of momentum and conservation of energy are usually taken as empirical law, but we can find a framework to view any conservation law as a result of a corresponding special symmetry.

1.1 Example: Minimising distance in $\mathbb{R}^n$

Recall that, given a curve $\Gamma \subset \mathbb{R}^2$ and a $C^1([0,1])$ parametrisation of said curve $\gamma\colon [0,1] \to \mathbb{R}^n$, the **arc length**, $\ell(\Gamma)$, of the curve is given by:

$$\ell(\Gamma) := \int_0^1 ||\gamma'(t)||\mathrm{d}t$$

We also require that the parametrisation γ is *regular*, that is, $\gamma'(t) \neq 0$ for all $t \in [0,1]$. Given two distinct points $a, b \in \mathbb{R}^n$, can we find the curve that minimises the arc length between them? Intuitively we can see the answer is a straight line, but it is a good exercise to try and prove this by thinking *variationally*. We adapt the proof from MA3K8: Variational Principles, Symmetry and Conservation Laws [1, pp. 1–2].

Theorem 1.1. *Let $a, b \in \mathbb{R}^n$ with $a \neq b$. The straight line joining a and b is the path that minimises the distance between them.*

Proof. The line L_{ab} connecting a and b can be parametrised by $\gamma(t) = (1-t)a + tb, t \in [0,1]$. The curve is differentiable on $(0,1)$, and we see that $\gamma'(t) = b - a$ and thus:

$$\ell(L_{ab}) = \int_0^1 ||\gamma'(t)||\mathrm{d}t = \int_0^1 ||b-a||\mathrm{d}t = ||b-a||$$

We now consider a family of regular paths, F, that are continuously differentiable on $[0,1]$ that have endpoints a and b. Formally:

$$F = \left\{\gamma \in C^1([0,1])\colon \gamma(0) = a, \gamma(1) = b, \gamma'(t) \neq 0 \ \forall t \in [0,1]\right\}$$

Pick arbitrary $\gamma \in F$ and let Γ be the corresponding curve, then:

$$\begin{aligned}
0 < ||b-a||^2 &= (b-a)\cdot(b-a) \\
&= (b-a)\cdot\left(\int_0^1 \gamma'(t)\mathrm{d}t\right) && \text{FTC II} \\
&\leq ||b-a|| \left\|\int_0^1 \gamma'(t)\mathrm{d}t\right\| && \text{Cauchy-Schwarz} \\
&\leq ||b-a|| \int_0^1 ||\gamma'(t)||\mathrm{d}t && \text{Triangle Inequality} \\
&= ||b-a||\ell(\Gamma)
\end{aligned}$$

So $\ell(L_{ab}) \leq \ell(\Gamma)$ for any Γ with parametrisation in F. □

The general process for solving that problem was not so different from what we would normally expect in an optimisation problem – we start with a function we want to minimise, and vary the argument(s) until some condition that implies minimality is achieved, typically a derivative or gradient being equal to 0. Here, we looked at a family of paths and showed that varying the path from anything other than a straight line increases arc-length. This introduces a subtle shift in perspective from points to paths, as our objective functions may now depend not only the points of evaluation, but on the paths taken to get there. We will see more of this in the next section. Notably, we also required an ansatz solution of a straight line in order to solve the problem. Although this works when the problems are visualisable and physically intuitive, a general framework for minimising or maximising these path-dependent functions is much more desirable, for the sake of generality.

2 Variational Principles

The following section will focus on building up some important machinery used for solving variational problems – that is, problems involving varying parameters for extremisation – by extending the notion of the derivative to other mathematical objects of interest, namely the *functional*.

2.1 Functionals and Extrema

To begin, we recall some of the following definitions:

Definition 2.1 (Global and Local extremum)**.** Let $f\colon \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$. We say the function f has a:

i *Local maximum* at $\mathbf{x_0} \in \Omega$ if there exists $\varepsilon > 0$ such that $f(\mathbf{x}) \leq f(\mathbf{x_0})$ whenever $\mathbf{x} \in \mathbb{B}(\mathbf{x_0}, \varepsilon) \cap \Omega$.

ii *Local minimum* at $\mathbf{x_0} \in \Omega$ if there exists $\varepsilon > 0$ such that $f(\mathbf{x}) \geq f(\mathbf{x_0})$ whenever $\mathbf{x} \in \mathbb{B}(\mathbf{x_0}, \varepsilon) \cap \Omega$.

These definitions match what we would expect in the single-variable case, and similarly we can use derivatives (where possible) to characterise necessary conditions for extrema.

Theorem 2.1. *Let $\Omega \subseteq \mathbb{R}^n$ be open. If the function $f\colon \Omega \to \mathbb{R}$ is differentiable and attains an extremum at $\mathbf{x_0} \in \Omega$, then:*

$$\partial_i f(\mathbf{x_0}) = 0 \ \forall i \in \{1, \dots, n\}$$

Where $\partial_i f(\mathbf{x_0})$ is the partial derivative of f with respect to its ith variable.

Proof. We adapt the proof from Spivak's *Calculus on Manifolds* [2, p. 27] Fix $i \in \{1, \dots, n\}$. We let $\mathbf{x_0} = (\mathbf{x_0}^{(0)}, \mathbf{x_0}^{(1)}, \dots, \mathbf{x_0}^{(n)})$ Consider the following single-variable function:

$$g_i(x) = (\mathbf{x_0}^{(0)}, \dots, x, \dots, \mathbf{x_0}^{(n)})$$

where x is the ith entry. Clearly, the function g_i attains an extremum at $x = \mathbf{x_0}^{(i)}$, and we can deduce via Rolle's Theorem that $g_i'\left(\mathbf{x_0}^{(i)}\right) = 0$. But this is simply the partial derivative of f with respect to its ith variable evaluated at $\mathbf{x_0}$, so $\partial_i f(\mathbf{x_0}) = 0$. Repeating the argument with suitably defined g_i for each $i \in \{1, \dots, n\}$ yields the required result. □

This provides us with a useful corollary:

Corollary 2.1. *Let $\Omega \subseteq \mathbb{R}^n$ be open. Then if the function $f\colon \Omega \to \mathbb{R}$ is differentiable on Ω and attains an extremum at $\mathbf{x_0} \in \Omega$, then $\partial_{\mathbf{v}} f(\mathbf{x_0}) = 0 \ \forall \mathbf{v} \in \mathbb{R}^n$*

Now we have a more general definition of extrema (and more general necessary conditions to match), we can define a new object of interest for variational problems: the functional.

Definition 2.2 (Functionals)**.** [3][p. 1] A real-valued function on a space of functions is called a functional.

We will focus on functionals that are definite integrals, that is, of the form $I\colon C^2([a,b]) \to \mathbb{R}$ given by:

$$I[y] = \int_a^b F\left(x, y(x), y'(x)\right) \mathrm{d}x \tag{2.1}$$

With $F\colon [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ twice continuously differentiable.

Searching for the function $y(x)$ that extremises $I[y]$ is a matter of considering all possible varied paths $\bar{y}$ that have fixed endpoints a, b and evaluating I along such paths by integrating. If we want to define this space of paths, it can help by quantifying their 'variation' from the optimal path (which we assume exists).

Definition 2.3 (Space of perturbartions)**.** [3][p. 29] Let $P := \{y \in C^2([a,b])\colon y(a) = y_a, y(b) = y_b\}$ denote the set of paths between $a, b \in \mathbb{R}$. Then we define the space of perturbations of P as:

$$H_P := \{\eta \in C^2([a,b])\colon \eta(a) = \eta(b) = 0\}$$

Remark. Any $\bar{y} \in P$ can be written as $y + \varepsilon\eta$ with $y \in P, \eta \in H_P, \varepsilon \neq 0$. To convince yourself of this, note that $C^2([a,b])$ is a vector space and work through the definition of H_P.

We can now write any path between a and b as a perturbation of some extremal path. We know that along the extremal path our functional should, naturally, be extremised, so any extremal path y must satisfy:

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} I[y + \varepsilon\eta]\bigg|_{\varepsilon=0} = 0 \tag{2.2}$$

The above expression should look familiar to the definition of the directional derivative:

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{x} + t\mathbf{v})\bigg|_{t=0}$$

Which provides an alternate, more abstract perspective: we want the directional derivative of the functional to be 0 in the 'direction' of all perturbations η. This also matches the extremum condition we proved in Theorem 2.1. We can now use Equation 2.2 combined with Equation 2.1 to derive an equation for the extremal path y that will be very useful for variational analysis.

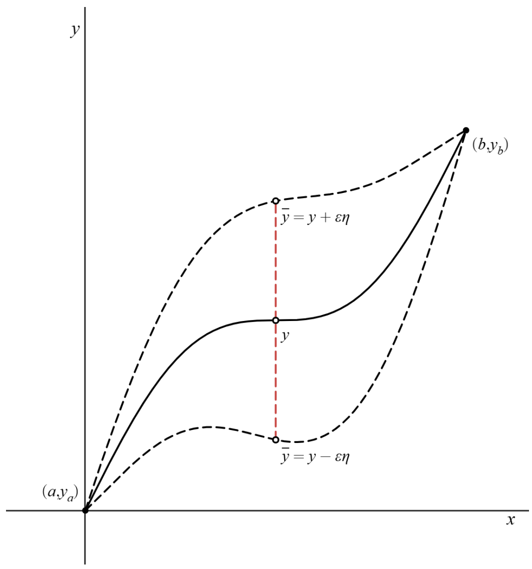

Figure 2.1: An example path, y, and two of its perturbations, $\bar{y} = y \pm \varepsilon\eta$

2.2 The Euler-Lagrange equation

We start with a lemma [4, p.30]:

Lemma 2.2 (Fundamental Lemma of the Calculus of Variations)**.** *Let $f\colon [a,b] \subset \mathbb{R} \to \mathbb{R}$ be continuous. Then if:*

$$\int_a^b f(x)\eta(x)\,dx = 0$$

for all $\eta \in C^1([a,b])$ *such that* $\eta(a) = \eta(b) = 0$*, then* $f(x) \equiv 0$.

Proof. [4, p.30] Suppose that $f(\bar{x}) \neq 0$ for some $x \in [a,b]$. By the continuity of f, there exists a sub-interval $[c,d] \subset [a,b]$ containing $\bar{x}$ such that f is nonzero and does not change sign across the interval. WLOG say f is positive on $[c,d]$. We construct the following 'bump' function:

$$\eta \in C^1([a,b]),\ \eta(x) = \begin{cases} (x-c)^2(x-d)^2 & x \in [c,d] \\ 0 & \text{else} \end{cases}$$

Then:

$$\int_a^b f(x)\eta(x)\mathrm{d}x > 0$$

Which is a contradiction. □

We also recall the following lemma:

Lemma 2.3 (Leibniz's Rule)**.** *Let the function* $f\colon \mathbb{R}^n \to \mathbb{R}$ *be* $C^1([a,b])$*. Then:*

$$\frac{d}{dx_i}\int_a^b f(x_1,\ldots,x_n)\,dx_j = \int_a^b \frac{\partial}{\partial x_i} f(x_1,\ldots,x_n)\,dx_j$$

for $1 \leq i,j \leq n$.

This is often referred to as 'differentiating under the integral sign'. Equipped with these results, we can evaluate expressions like Equation 2.2 for functionals that are definite integrals. Let $\bar{y}$ denote a perturbation of y, that is, $\bar{y} = y + \varepsilon\eta$ for some $y \in P,\ \eta \in H_p,\ \varepsilon \in \mathbb{R}$. Then:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\varepsilon} I[y+\varepsilon\eta] &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int_a^b F\left(x, y(x)+\varepsilon\eta(x), y'(x)+\varepsilon\eta'(x)\right)\mathrm{d}x \\ &= \int_a^b \frac{\partial}{\partial\varepsilon} F\left(x, y(x)+\varepsilon\eta(x), y'(x)+\varepsilon\eta'(x)\right)\mathrm{d}x && \text{By Lemma 2.3} \\ &= \int_a^b \left(\cancel{\frac{\partial F}{\partial x}\frac{\partial x}{\partial \varepsilon}} + \frac{\partial F}{\partial \bar{y}}\frac{\partial \bar{y}}{\partial \varepsilon} + \frac{\partial F}{\partial \bar{y}'}\frac{\partial \bar{y}'}{\partial \varepsilon}\right)\mathrm{d}x && \text{Multivariable Chain Rule} \\ &= \int_a^b \left(\frac{\partial F}{\partial \bar{y}}\eta(x) + \frac{\partial F}{\partial \bar{y}'}\eta'(x)\right)\mathrm{d}x \end{aligned}$$

It is important to note that we assume independence of $\bar{y}$ and $\bar{y}'$, and we have computed $\frac{\partial F}{\partial \epsilon}$ by differentiating with respect to the arguments of F, not with respect to functions. Integrating by parts on the right-hand term, we see that:

$$\begin{aligned} \int_a^b \frac{\partial F}{\partial \bar{y}'}\eta'(x) &= \frac{\partial F}{\partial \bar{y}'}\eta(x)\Big|_{x=a}^{x=b} - \int_a^b \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial \bar{y}'}\eta(x)\mathrm{d}x \\ &= -\int_a^b \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial \bar{y}'}\eta(x)\mathrm{d}x \end{aligned}$$

Where we have used that $\eta \in H_P$ so $\eta(a) = \eta(b) = 0$. Returning to the previous expression:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\varepsilon} I[y+\varepsilon\eta] &= \int_a^b \left(\frac{\partial F}{\partial \bar{y}}\eta(x) - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial \bar{y}'}\eta(x)\right)\mathrm{d}x \\ &= \int_a^b \left(\frac{\partial F}{\partial \bar{y}} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial \bar{y}'}\right)\eta(x)\mathrm{d}x \end{aligned}$$

Noting that we have $\eta \in H_P \subset C^2([a,b]) \subset C^1([a,b])$ and F twice continuously differentiable, so we can apply Lemma 2.2. We also set $\varepsilon = 0$ and set the entire derivative to 0 as in Equation 2.2. This forces $\bar{y} = y$, and thus any extremal y must satisfy:

$$\frac{\partial F}{\partial y} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'} \tag{2.3}$$

Which is known as the *Euler-Lagrange equation.*

This is an extremely powerful tool for solving variational problems. Rather than having to make educated guesses based on intuition or observations, we have instead have a necessary condition for functional extremisation. Determining the nature of any subsequently derived extrema involves calculating higher 'derivatives' of the functional, but we will focus on first derivatives for now. We could also have arrived to this same equation by method of a series expansion, which allows us to simultaneously define a quantity known as the 'first variation', and also arrive at another condition for extremisation. Say $\bar{y} = y + \varepsilon\eta$, then for ε sufficiently small, we can apply Taylor's theorem [3, pp.29–30]:

$$\begin{aligned} F(x, \bar{y}, \bar{y}') &= F(x, y + \varepsilon\eta, y' + \varepsilon\eta') \\ &= F(x, y, y') + \varepsilon\left(\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta'\right) + \mathcal{O}(\varepsilon^2) \end{aligned}$$

Then:

$$I[\bar{y}] - I[y] = \underbrace{\varepsilon\int_a^b \left(\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta'\right)\mathrm{d}x}_{\delta I(y,\eta)} + \mathcal{O}(\varepsilon^2)$$

Where:

$$\delta I(y,\eta) := \int_a^b \left(\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta'\right)\mathrm{d}x$$

is called the *first variation* of I. Note that $\mathcal{O}(\varepsilon^2)$ is regarded as $\varepsilon \to 0$. For extremisation we require that that this quantity vanishes, again leading to the Euler-Lagrange equation as a necessary condition. The difference in functional outputs being of order ε^2 when the variation between paths is of order ε is a very useful formulation we will use more later when examining local symmetries.

2.2.1 Example: Revisiting distance with $n = 2$

Let us revisit the example presented in Section 1.1 of minimising distance in $\mathbb{R}^n$, setting $n = 2$ for simplicity. Armed with the Euler-Lagrange equation, we will show that considering a space of functions in the form $y(x)$ and solving the relevant differential equations will also yield a straight-line geodesic. Consider the functional $\ell\colon C^2([a,b]) \to \mathbb{R}$ with $a, b \in \mathbb{R}$ given by:

$$\ell[y] = \int_a^b \sqrt{1 + (y'(x))^2}\,\mathrm{d}x$$

This is an equivalent formulation of the arc length for a function $y(x)$ with endpoints $x = a$ and $x = b$. We calculate the following partial derivatives:

$$\begin{aligned} &F(x, y, y') = \sqrt{1 + (y'(x))^2} \\ &\frac{\partial F}{\partial y} = 0,\ \frac{\partial F}{\partial y'} = \frac{1}{2\sqrt{1 + (y'(x))^2}} \cdot 2y'(x) = \frac{y'(x)}{\sqrt{1 + (y'(x))^2}} \end{aligned}$$

Applying Euler-Lagrange:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\frac{y'(x)}{\sqrt{1+(y'(x))^2}} = 0 \implies \frac{y'(x)}{\sqrt{1+(y'(x))^2}} &= C_1,\ C_1 \in \mathbb{R} \\
\implies y'(x) &= \sqrt{\frac{C_1^2}{1-C_1^2}}, && |C_1| < 1 \\
\implies y(x) &= \left(\sqrt{\frac{C_1^2}{1-C_1^2}}\right)x + C_2, && |C_1| < 1, C_2 \in \mathbb{R}
\end{aligned}$$

We can evaluate $y(x)$ at $x = a$, $x = b$ and $x = 0$ to obtain:

$$y(x) = \left(\frac{y(b) - y(a)}{b - a}\right)x + y(0)$$

Which is clearly the straight line that passes through $(a, y(a))$ and $(b, y(b))$, as we expected. It is important to note that satisfying the Euler-Lagrange equations is only a necessary condition for extremisation, not a sufficient condition, and so other tools [1] would then be used to prove that the derived extremal path is minimal.

2.2.2 Endpoint conditions

Up until now we have been considering variational problems that over a space of paths that have *fixed endpoints*, but it is possible to analyse and solve problems where we search for an extremal curve where both independent and dependent coordinate endpoints are allowed to vary. This is a useful generalisation to seek, as some functionals that exhibit *spatial symmetries* can have extremal paths whose start and endpoint do not matter. We follow the arguments in Brunt's *The Calculus of Variations* closely in this subsection [3][pp.144-148]. Consider a real-valued functional of the form:

$$I[y] = \int_a^b F(x, y, y')\ \mathrm{d}x$$

Where F is smooth. We now change the integration limits such that they may vary with the pertubed path. That is, for $\bar{y} = y + \varepsilon\eta$:

$$I[\bar{y}] = \int_{\bar{a}}^{\bar{b}} F(x, \bar{y}, \bar{y}')\ \mathrm{d}x$$

Where $\eta \in C^2([\min(a, \bar{a}), \max(b, \bar{b})])$, so that we may work over a common interval. We can also write the differences between the path endpoints of y and $\bar{y}$ as multiples of ε:

$$\begin{gathered}
\bar{a} = a + \varepsilon X_0, \bar{b} = b + \varepsilon X_1 \\
\bar{y}_{\bar{a}} = y_a + \varepsilon Y_0, \bar{y}_{\bar{b}} = y_b + \varepsilon Y_1
\end{gathered}$$

We can compute the difference in functional outputs directly, and observe what necessary conditions an extremal path would satisfy, just as in the previous section.

[1] Just as the first variation corresponds to the first derivative, we can define and analyse *second variations* of functionals that reveal more information about the nature of extremal paths

$$
\begin{aligned}
I[\bar{y}] - I[y] &= \int_{\bar{a}}^{\bar{b}} F(x, \bar{y}, \bar{y}')\mathrm{d}x - \int_a^b F(x, y, y')\mathrm{d}x \\
&= \int_{a+\varepsilon X_0}^{b+\varepsilon X_1} F(x, \bar{y}, \bar{y}')\mathrm{d}x - \int_a^b F(x, y, y')\mathrm{d}x \\
&= \int_a^b \left(F(x, \bar{y}, \bar{y}') - F(x, y, y')\right) \mathrm{d}x + \int_b^{b+\varepsilon X_1} F(x, \bar{y}, \bar{y}')\mathrm{d}x \\
&\quad - \int_a^{a+\varepsilon X_0} F(x, \bar{y}, \bar{y}')\mathrm{d}x
\end{aligned}
$$

For small ε, which we examine as we consider $\varepsilon \to 0$ when $\bar{y} \to y$, we may take the following expansions to $\mathcal{O}(\varepsilon^2)$:

$$
\begin{aligned}
\int_b^{b+\varepsilon X_1} F(x, \bar{y}, \bar{y}')\mathrm{d}x &= \varepsilon X_1 F(x, y, \bar{y}')\Big|_{x=b} + \mathcal{O}(\varepsilon^2) \\
\int_a^{a+\varepsilon X_0} F(x, \bar{y}, \bar{y}')\mathrm{d}x &= \varepsilon X_0 F(x, y, \bar{y}')\Big|_{x=a} + \mathcal{O}(\varepsilon^2)
\end{aligned}
$$

Using Equation 2.2 and integrating by parts yields:

$$
\begin{aligned}
I[\bar{y}] - I[y] = \varepsilon \Bigg(& \eta \frac{\partial F}{\partial y'}\Big|_{x=a}^{x=b} + \int_a^b \eta \left(\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'} \right) \mathrm{d}x \\
& + X_1 F(x, y, y')\Big|_{x=b} - X_0 F(x, y, y')\Big|_{x=a} \Bigg) \\
& + \mathcal{O}(\varepsilon^2)
\end{aligned}
$$

We can then use the path endpoint conditions that characterise $\bar{y}$ and y to arrive at the following relation:

$$
\begin{aligned}
I[\bar{y}] - I[y] = \varepsilon \Bigg(& \eta \frac{\partial F}{\partial y'}\Big|_{x=a}^{x=b} + \int_a^b \eta \left(\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'} \right) \mathrm{d}x \\
& + Y_1 \frac{\partial F}{\partial y'}\Big|_{x=b} - Y_0 \frac{\partial F}{\partial y'}\Big|_{x=a} \\
& + X_1 \left(F - y'\frac{\partial F}{\partial y'} \right)\Big|_{x=b} - X_0 \left(F - y'\frac{\partial F}{\partial y'} \right)\Big|_{x=a} \Bigg) \\
& + \mathcal{O}(\varepsilon^2)
\end{aligned}
$$

We argue as when deriving the Euler-Lagrange equation that we require all terms of order ε to vanish for all variations η. The critical insight here is that as this must hold true for **all** variations η, it must hold when η belongs to the corresponding space of perturbations of the extremal, H_P. This means that the extremal path must again satisfy the Euler-Lagrange equations, and additionally:

$$
\left(\frac{\partial F}{\partial y'} \Delta Y(x) - \left(y' \frac{\partial F}{\partial y'} - F \right) \Delta X(x) \right) \Big|_{x=a}^{x=b} = 0
$$

Where:

$$
\begin{aligned}
\Delta Y(a) = Y_0,\ \Delta Y(b) = Y_1 \\
\Delta X(a) = X_0, \Delta X(b) = X_1
\end{aligned}
$$

simply encode the values attained at the boundaries a, b. We can simplify this condition further by introducing two more quantities:

$$p = \frac{\partial F}{\partial y'} \qquad \text{“Canonical Momentum”}$$
$$H = y'p - F \qquad \text{“The Hamiltonian”}$$

Which yields:

$$(p\Delta Y(x) - H\Delta X(x))\Big|_{x=a}^{x=b} = 0 \tag{2.4}$$

Which we will henceforth refer to as the *Free-endpoint condition.* This condition is helpful for analysing problems where we still desire extremisation but cannot make any further assumptions about the boundary conditions a solution must satisfy, in the same way one might solve a differential equation to find a general family of solutions before turning it into an initial value problem. Referring back to the example presented in Section 1.1 and Section 2.2.1, if we had not prescribed any boundary conditions $y(a), y(b)$, then the entire family of straight lines in $\mathbb{R}^2$ would be extremal paths.

2.2.3 Extension to generalised lagrangians

Previously we looked at functionals of the form $I[y]$ where $y\colon \mathbb{R} \to \mathbb{R}$, but we can generalise the Euler-Lagrange equation to handle vector-valued functions of a single variable too. We will let $C^k([a,b],\mathbb{R}^n)$ be the set of functions $\mathbf{q}\colon [a,b] \to \mathbb{R}^n$ that have k continuous derivatives. The space of perturbations P for the general case is defined similarly to Definition 2.3:

$$P \coloneqq \{\mathbf{q} \in C^2([a,b],\mathbb{R}^n)\colon \mathbf{q}(a) = \mathbf{q}_a, \mathbf{q}(b) = \mathbf{q}_b\}, H_P \coloneqq \{\eta \in C^2([a,b],\mathbb{R}^n)\colon \eta(a) = \eta(b) = \mathbf{0}\}$$

Our updated functional thus looks like:

$$I[\mathbf{q}] = \int_a^b \mathcal{L}(t, \mathbf{q}, \dot{\mathbf{q}})\mathrm{d}t$$

We now call the twice continuously differentiable function $\mathcal{L}\colon \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ the *lagrangian* of I. We can repeat the steps in 2.2 in this updated case:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\varepsilon} I[\mathbf{q} + \varepsilon\eta] &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \int_a^b \mathcal{L}(t, \mathbf{q} + \varepsilon\eta, \dot{\mathbf{q}} + \varepsilon\dot{\eta})\mathrm{d}t \\ &= \int_a^b \frac{\partial}{\partial\varepsilon} \mathcal{L}(t, \mathbf{q} + \varepsilon\eta, \dot{\mathbf{q}} + \varepsilon\dot{\eta})\mathrm{d}t \\ &= \int_a^b \sum_{i=1}^n \left(\frac{\partial \mathcal{L}}{\partial q_i}\eta + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\dot{\eta} \right) \mathrm{d}t \end{aligned} \tag{2.5}$$

We use an insight from Brunt's *The Calculus of Variations*[3][pp.61–62] to conclude the derivation. Consider a family of subsets of H_P given by:

$$H_P^{(i)} = \{(0, \dots, \eta_i, \dots, 0) \in H_P\},\ i \in \{1, \dots, n\}$$

As before we desire our 'derivative' to vanish for any perturbation η that we pick. If $\eta \in H_P^{(i)}$ then Equation 2.5 simplifies to:

$$\int_a^b \left(\frac{\partial \mathcal{L}}{\partial q_i}\eta_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\dot{\eta}_i \right) \mathrm{d}t$$

Which we know vanishes when the following Euler-Lagrange equation is satisfied:

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

Iterating over $i \in \{1, \dots, n\}$ provides a system of Euler-Lagrange equations that must be satisfied for extremisation. Just as we can extend the Euler-Lagrange equation into a system of equations, we can do the same for the Free-endpoint condition. The proof is omitted, but any extremal path $\mathbf{q}(t)$ with no further boundary conditions prescribed will satisfy:

$$\sum_{i=1}^{n} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \Delta Q_i(t) - \left(\sum_{i=1}^{n} \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \mathcal{L} \right) \Delta T(t) = 0$$

Which, as before, can be simplified using notions of the hamiltonian and canonical momentum:

$$\sum_{i=1}^{n} p_i \Delta Q_i(t) - H \Delta T(t) = 0 \tag{2.6}$$

Where:

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}, H = \sum_{i=1}^{n} \dot{q}_i p_i - \mathcal{L} \tag{2.7}$$

Are the *generalised* canonical momenta and hamiltonian for the system.

3 Invariance and Symmetry

3.1 Symmetries of a lagrangian

Having explored standalone extremisation sufficiently, we will now turn our attention to the notion of *symmetry* and what it can imply for systems that can be analysed with variational principles. Intuitively speaking, a dynamic or physical system is governed by a corresponding set of laws, and these laws are recoverable from the system's lagrangian. We think of symmetries as transformations of the lagrangian that leave the functional or lagrangian itself unchanged. We can formalise this concept as follows:

Definition 3.1 (One-parameter symmetry)**.** [3, pp.202–204] Consider a real-valued functional I with parametric lagrangian $\mathcal{L} \in C^1(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n)$ of the form $\mathcal{L}(t, \mathbf{q}, \dot{\mathbf{q}})$. Then we define the following transformations:

$$T := \theta(t, \mathbf{q}; \varepsilon), \ \mathbf{Q} := \psi(t, \mathbf{q}; \varepsilon)$$

Where θ and ψ are smooth functions of $\mathbf{q}$ and ε such that:

$$\theta(t, \mathbf{q}; 0) = t, \ \psi(t, \mathbf{q}; 0) = \mathbf{q}$$

These transformations are called *symmetries* of $\mathcal{L}$ (or I) if:

$$I[\mathbf{q}] := \int_a^b \mathcal{L}(t, \mathbf{q}(t), \dot{\mathbf{q}}(t)) \mathrm{d}t = \int_{\theta(a,\mathbf{q}(a);\varepsilon)}^{\theta(b,\mathbf{q}(b);\varepsilon)} \mathcal{L}(T, \mathbf{Q}(T), \dot{\mathbf{Q}}(T)) \mathrm{d}T$$

For all smooth functions $\mathbf{q} \colon \mathbb{R} \to \mathbb{R}^n$ and ε sufficiently small so that θ and ψ are invertible.

Here $\dot{\mathbf{Q}} = \frac{\mathrm{d}}{\mathrm{d}T}\mathbf{Q}$. So each symmetry takes a parameter ε, transforms all the relevant coordinates and leaves the functional unchanged.

Example 3.1 (Kepler's Planetary Motion)**.** [3][p.206] Set $a, b \in \mathbb{R}$. Define:

$$\mathcal{L}(t, \mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2} m \left(\dot{q_1}^2 + \dot{q_2}^2 \right) + \frac{GMm}{\sqrt{q_1^2 + q_2^2}}$$

For some $G, M, m \in \mathbb{R}$. This lagrangian is of the form $\mathcal{L} = T - V$, where T models the *kinetic* energy of the physical system and V the *potential* energy. Lagrangians of this form are common in mathematical physics, as the equations of motion of a physical system can be recovered from how its energy changes with respect to some configuration space (this is discussed in greater detail in Section 4 under Hamilton's Principle). Specifically, this lagrangian models planetary motion of a celestial body of mass m about a larger body of mass $M \gg m$. Consider the transformations:

$$T = \theta(t, \mathbf{q}; \varepsilon) = t \tag{3.1}$$
$$Q_1 = \psi_1(t, \mathbf{q}; \varepsilon) = q_1 \cos(\varepsilon) + q_2 \sin(\varepsilon) \tag{3.2}$$
$$Q_2 = \psi_2(t, \mathbf{q}; \varepsilon) = -q_1 \sin(\varepsilon) + q_2 \cos(\varepsilon) \tag{3.3}$$

This represents a spatial rotation of the coordinates $\mathbf{q}$ by an angle ε. The transformation is smooth as both sine and cosine are smooth functions, and the transformations are invertible for any $\varepsilon \in \mathbb{R}$, as is standard with rotations (recall $\mathbf{SO}(n)$ is a group). We have that the operators $\frac{\mathrm{d}}{\mathrm{d}t}$ and $\frac{\mathrm{d}}{\mathrm{d}T}$ are equal by Equation 3.1, and so:

$$\begin{aligned}\dot{Q}_1 &= \dot{q}_1 \cos(\varepsilon) + \dot{q}_2 \sin(\varepsilon)\\ \dot{Q}_2 &= -\dot{q}_1 \sin(\varepsilon) + \dot{q}_2 \cos(\varepsilon)\end{aligned}$$

Substituting these into the lagrangian, we get:

$$\mathcal{L}(T, \mathbf{Q}, \dot{\mathbf{Q}}) = \frac{1}{2} m \left(\dot{Q}_1{}^2 + \dot{Q}_2{}^2\right) + \frac{GMm}{\sqrt{Q_1^2 + Q_2^2}}$$

Simplifying the kinetic term:

$$\begin{aligned}\dot{Q}_1{}^2 + \dot{Q}_2{}^2 &= \dot{q}_1{}^2 \cos^2(\varepsilon) + 2\dot{q}_1\dot{q}_2 \sin(\varepsilon)\cos(\varepsilon) + \dot{q}_2{}^2 \sin^2(\varepsilon)\\ &\quad + \dot{q}_1{}^2 \sin^2(\varepsilon) - 2\dot{q}_1\dot{q}_2 \sin(\varepsilon)\cos(\varepsilon) + \dot{q}_2{}^2 \cos^2(\varepsilon)\\ &= \dot{q}_1{}^2 \left(\sin^2(\varepsilon) + \cos^2(\varepsilon)\right) + \dot{q}_2{}^2 \left(\sin^2(\varepsilon) + \cos^2(\varepsilon)\right)\\ &= \dot{q}_1{}^2 + \dot{q}_2{}^2\end{aligned}$$

And similarly for the denominator in the potential term:

$$\begin{aligned}Q_1^2 + Q_2^2 &= q_1^2 \cos^2(\varepsilon) + 2q_1q_2 \sin(\varepsilon)\cos(\varepsilon) + q_2^2 \sin^2(\varepsilon)\\ &\quad + q_1^2 \sin^2(\varepsilon) - 2q_1q_2 \sin(\varepsilon)\cos(\varepsilon) + q_2^2 \cos^2(\varepsilon)\\ &= q_1^2 \left(\cos^2(\varepsilon) + \sin^2(\varepsilon)\right) + q_2^2 \left(\cos^2(\varepsilon) + \sin^2(\varepsilon)\right)\\ &= q_1^2 + q_2^2\end{aligned}$$

Then:

$$\begin{aligned}\mathcal{L}(T, \mathbf{Q}, \dot{\mathbf{Q}}) &= \frac{1}{2} m \left(\dot{q}_1{}^2 + \dot{q}_2{}^2\right) + \frac{K}{\sqrt{q_1^2 + q_2^2}}\\ &= \mathcal{L}(t, \mathbf{q}, \dot{\mathbf{q}})\end{aligned}$$

We also have $\theta(a, \mathbf{q}(a); \varepsilon) = a$ and $\theta(b, \mathbf{q}(b); \varepsilon) = b$, and thus:

$$\int_a^b \mathcal{L}(t, \mathbf{q}, \dot{\mathbf{q}})\mathrm{d}t = \int_{\theta(a,\mathbf{q}(a);\varepsilon)}^{\theta(b,\mathbf{q}(b);\varepsilon)} \mathcal{L}(T, \mathbf{Q}, \dot{\mathbf{Q}})\mathrm{d}T$$

So T, Q_1 and Q_2 are symmetries of $\mathcal{L}$. Physically, this result suggests that the laws governing planetary motion do not change if you rotate your frame of reference in the same axis as that of the rotation. Results like this may seem obvious, but isotropy of space is a very useful assumption to be able to make in mathematical modelling and physics. [2]

[2]The Michelson-Morley experiment performed in 1887 was an attempt to prove the existence of the Luminiferous Aether, a 'wind' that would act as the propagation medium for light. Instead it was proven to not exist, which allowed theories like Einstein's Special Relativity to develop.

3.2 Extremal invariance

The ideas and order of information in this subsection are again based on Brunt's *The Calculus of Variations* [3][pp. 44–46], with additional calculations and motivation added for completeness. A natural question to ask now would be how does the extremal path for a functional change with respect to its symmetries? That is, does an extremal path remain extremal under certain choices of coordinates? We will consider a more general family of transformations than the one-parameter symmetries defined in Equation 3.1 but focus on the 2-dimensional case for clarity. We recall that coordinate transformations $x, y \colon \mathbb{R}^2 \to \mathbb{R}$ with form:

$$x = x(u, v), y = y(u, v) \tag{3.4}$$

have an associated property known as the *jacobian determinant*. We assume that x and y are smooth transformations – they have continuous partial derivatives with respect to the variables u and v. The jacobian determinant of such a transformation is given by:

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{pmatrix}$$

We recall that a transformation is *nonsingular* if:

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| \neq 0 \ \forall (u,v) \in \mathbb{R}^2$$

Notably, a transformation is nonsingular if and **only if** it is invertible. Let I be a functional of the form:

$$I[y] = \int_a^b \mathcal{L}(x, y, y')\mathrm{d}x$$

and P be the corresponding space of paths as in Definition 2.3. We can use the multivariable chain rule along with integration by substitution to observe how I changes when transforming the xy-plane to the uv-plane. Note that in this formulation we have assumed y to be a function of x, and thus we also assume v to be a function of u.

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}u} &= \frac{\partial y}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}u} + \frac{\partial y}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}u} \\ &= \frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}v'\end{aligned}$$

And similarly for $x(u, v)$:

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}u} &= \frac{\partial x}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}u} + \frac{\partial x}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}u} \\ &= \frac{\partial x}{\partial u} + \frac{\partial x}{\partial v}v'\end{aligned}$$

Then applying the chain rule once more yields:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} \\ &= \frac{y_u + y_v v'}{x_u + x_v v'}\end{aligned}$$

Where x_u, x_v and y_u, y_v are the partial derivatives of x and y with respect to u and v, respectively. We also perform an integral substitution as follows:

$$\begin{aligned}\mathrm{d}x &= \frac{\mathrm{d}x}{\mathrm{d}u}\mathrm{d}u \\ &= (x_u + x_v v')\mathrm{d}u\end{aligned}$$

Substituting these expressions into $I[y]$ yields:

$$\begin{aligned} I[y] &= \int_a^b \mathcal{L}(x,y,y')\mathrm{d}x \\ &= \int_{u_a}^{u_b} \mathcal{L}\left(x(u,v), y(u,v), \frac{y_u + y_v v'}{x_u + x_v v'}\right)(x_u + x_v v')\mathrm{d}u \\ &= \int_{u_a}^{u_b} \tilde{\mathcal{L}}(u,v,v')\mathrm{d}u \\ &= \tilde{I}[v] \end{aligned}$$

With u_0, u_1, v_0, v_1 all satisfying the system:

$$\begin{aligned} v(u_0) &= v_0 & a &= x(u_0, v_0) & y_a &= y(u_0, v_0) \\ v(u_1) &= v_1 & b &= x(u_1, v_1) & y_b &= y(u_1, v_1) \end{aligned}$$

We also define a new space of paths for v as in Definition 2.3:

$$\tilde{P} := \{v \in C^2([u_0, u_1]) \colon v(u_0) = v_0, v(u_1) = v_1\}$$

Now we have a way to transform between $I[y]$ and $\tilde{I}[v]$ we are in a better position to ask the following question: If $y(x) \in P$ is an extremal for I and $v(u)$ is the transformation of y under Equation 3.4, then is $v(u)$ extremal for $\tilde{I}$?

Theorem 3.1. *[3][p. 45] Let $y \in P, v \in \tilde{P}$ be two real-valued functions on $\mathbb{R}^2$ satisfying smooth, nonsingular transformations of the form:*

$$x = x(u,v), y = y(u,v)$$

and let $I[y], \tilde{I}[v]$ be the corresponding functionals defined over the space of paths for y and v respectively. Then y is an extremal of $I[y]$ if and only if v is an extremal of $\tilde{I}[v]$.

Proof. [3][pp. 45–46] Suppose $v \in \tilde{P}$ is extremal for $\tilde{I}[v]$. Then v satisfies:

$$\frac{\mathrm{d}}{\mathrm{d}u}\frac{\partial \tilde{\mathcal{L}}}{\partial v'} - \frac{\partial \tilde{\mathcal{L}}}{\partial v} = 0 \tag{3.5}$$

Where:

$$\tilde{\mathcal{L}}(u,v,v') = \mathcal{L}\left(x(u,v), y(u,v), \frac{y_u + y_v v'}{x_u + x_v v'}\right)(x_u + x_v v')$$

We compute the partial derivative with respect to v' using the product and chain rule.

$$\begin{aligned} \frac{\partial \tilde{\mathcal{L}}}{\partial v'} &= \frac{\partial}{\partial v'}\left[\mathcal{L}\left(x(u,v), y(u,v), \frac{y_u + y_v v'}{x_u + x_v v'}\right)\right](x_u + x_v v') \\ &\quad + \mathcal{L}\left(x(u,v), y(u,v), \frac{y_u + y_v v'}{x_u + x_v v'}\right)\frac{\partial}{\partial v'}\left[(x_u + x_v v')\right] \\ &= \left(\frac{\partial \mathcal{L}}{\partial x}\cancel{\frac{\partial x}{\partial v'}} + \frac{\partial \mathcal{L}}{\partial y}\cancel{\frac{\partial y}{\partial v'}} + \frac{\partial \mathcal{L}}{\partial y'}\frac{\partial}{\partial v'}\left[\frac{y_u + y_v v'}{x_u + x_v v'}\right]\right)(x_u + x_v v') + x_v \mathcal{L} \\ &= \frac{\partial \mathcal{L}}{\partial y'}(x_u + x_v v')\frac{\partial}{\partial v'}\left[\frac{y_u + y_v v'}{x_u + x_v v'}\right] \end{aligned}$$

And similarly for v:

$$\begin{aligned} \frac{\partial \tilde{\mathcal{L}}}{\partial v} &= \frac{\partial}{\partial v}\left[\mathcal{L}\left(x(u,v), y(u,v), \frac{y_u + y_v v'}{x_u + x_v v'}\right)\right](x_u + x_v v') \\ &\quad + \mathcal{L}\left(x(u,v), y(u,v), \frac{y_u + y_v v'}{x_u + x_v v'}\right)\frac{\partial}{\partial v}\left[(x_u + x_v v')\right] \\ &= \left(\frac{\partial \mathcal{L}}{\partial x}x_v + \frac{\partial \mathcal{L}}{\partial y}y_v + \frac{\partial \mathcal{L}}{\partial y'}\frac{\partial}{\partial v}\left[\frac{y_u + y_v v'}{x_u + x_v v'}\right]\right)(x_u + x_v v') + \mathcal{L}\frac{\partial}{\partial v}\left[x_u + x_v v'\right] \end{aligned}$$

After some extensive manipulation, it can be shown that:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}u}\frac{\partial\tilde{\mathcal{L}}}{\partial v'}-\frac{\partial\tilde{\mathcal{L}}}{\partial v}&=(x_u y_v - x_v y_u)\left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial\mathcal{L}}{\partial y'}-\frac{\partial\mathcal{L}}{\partial y}\right)\\&=\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial\mathcal{L}}{\partial y'}-\frac{\partial\mathcal{L}}{\partial y}\right)\end{aligned}$$

But by Equation 3.5, the left hand side is zero. We also know the jacobian determinant for the transformation is nonzero as the transformation is nonsingular, so we must have:

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial\mathcal{L}}{\partial y'}-\frac{\partial\mathcal{L}}{\partial y}\right)=0$$

which implies y is extremal for $I[y]$. Inverting the transformations proves the converse, and we are done. □

As the one-parameter symmetries introduced in Definition 3.1 are smooth and invertible, they also support the 'invariance' of the Euler-Lagrange equation. This allows us to analyse the lagrangian of a system in any sensible choice of coordinates, which is helpful for solving problems that exhibit their own symmetries – Example 3.1 could have been analysed with a lagrangian based on polar coordinates rather than cartesian coordinates to exploit the rotational symmetry of circular or elliptical orbits for example. So far, these one-parameter symmetries have conserved both the lagrangian and its corresponding Euler-Lagrange equation(s), which suggests a correspondence between symmetry and conservation as concepts. This is the main idea behind *Noether's Theorem*, that every continuous, differentiable symmetry of a system has a corresponding conservation law. That is to say, induces a quantity that is stationary over time. We will spend the next section formalising intuition behind conserved quantities and proving their induction from lagrangian symmetries.

4 Noether's Theorem

We begin with a formal definition of a conserved quantity:

Definition 4.1 (Conserved Quantity)**.** Let $\mathcal{L}\colon \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$ of the form $\mathcal{L}(t,\mathbf{q}(t),\dot{\mathbf{q}}(t))$ be the lagrangian of some corresponding system with variables $t,\mathbf{q},\dot{\mathbf{q}}$. Let $Q^* \in C^1([a,b]\times\mathbb{R}^n\times\mathbb{R}^n)$ be a function of the same system of variables. Then if:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[Q^*(t,\mathbf{q},\dot{\mathbf{q}})\right]=0\ \forall t\in[a,b]$$

then Q^* is called a conserved quantity.

We see that this condition is equivalent to the quantity Q^* being constant across the interval $[a,b]$ by the Mean Value Theorem. Now we have a concrete condition in mind when studying conservation, we can focus on building up tools to prove its correspondence to symmetry.

Definition 4.2 (Infinitesimal Generators)**.** [3][p. 208] Let $\theta(t,\mathbf{q};\varepsilon)$ be a one-parameter symmetry for a lagrangian with the form $\mathcal{L}(t,\mathbf{q},\dot{\mathbf{q}})$. Then we call:

$$\zeta(t,\mathbf{q})=\left.\frac{\partial\theta}{\partial\varepsilon}\right|_{(t,\mathbf{q};0)}$$

The *infinitesimal generator* of θ.

The exact role this quantity plays becomes apparent when applying Taylor's theorem to the corresponding transformation, but they can be thought of as a manner of approximating a transformation at the first order. We expect transformations that are *not* symmetries to produce a difference in functional output, and this difference can be characterised by the transformation's infinitesimal generators. Equipped with these definitions, we are now ready to state and prove Noether's Theorem.

Theorem 4.1 (Noether's Theorem for time-dependent Lagrangians). *[3][pp. 210–211] Suppose a twice continuously differentiable lagrangian $\mathcal{L}\colon [a,b]\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$ of the form $\mathcal{L}(t,\mathbf{q}(t),\dot{\mathbf{q}}(t))$ has symmetries with infinitesimal generators ζ,η. Define:*

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}, H = \sum_{i=1}^{n} p_i \dot{q}_i - \mathcal{L}$$

Then along any $\mathbf{q}(t)$ that extremises:

$$I[\mathbf{q}] = \int_a^b \mathcal{L}(t,\mathbf{q},\dot{\mathbf{q}})\mathrm{d}t$$

the quantity

$$\sum_{i=1}^{n} p_i\eta_i - H\zeta$$

is conserved.

Proof. [3][p. 209] We focus on the case of $n=1$ for clarity. Consider now a functional of the form:

$$I[y] = \int_\alpha^\beta \mathcal{L}(x,y,y')\mathrm{d}x$$

Where $[\alpha,\beta]\subseteq[a,b]$. Define the following one-parameter symmetries:

$$X \coloneqq \theta(x,y;\varepsilon), Y \coloneqq \psi(x,y;\varepsilon)$$

Using Definition 4.2 and taking $|\varepsilon|$ sufficiently small, we can make a Taylor expansion:

$$X = \theta(x,y;0) + \varepsilon\frac{\partial\theta}{\partial\varepsilon}\Big|_{(x,y;0)} + \mathcal{O}(\varepsilon^2) = \theta(x,y;0) + \varepsilon\zeta + \mathcal{O}(\varepsilon^2) = x + \varepsilon X_0 \tag{4.1}$$

$$Y = \psi(x,y;0) + \varepsilon\frac{\partial\psi}{\partial\varepsilon}\Big|_{(x,y;0)} + \mathcal{O}(\varepsilon^2) = \psi(x,y;0) + \varepsilon\eta + \mathcal{O}(\varepsilon^2) = y + \varepsilon Y_0 \tag{4.2}$$

Let $\alpha_\varepsilon = \theta(\alpha,y(\alpha);\varepsilon)$ and $\beta_\varepsilon = \psi(\beta,y(\beta);\varepsilon)$. Supposing $I[y]$ is stationary at y is equivalent to having a difference of functional outputs over any space of paths being $\mathcal{O}(\varepsilon^2)$, as in Section 2.2 when deriving the Euler-Lagrange equation. This is a *weaker* condition than what we have for a variational symmetry, which is no difference in functional output. Computing directly we have:

$$\begin{aligned} I[Y]-I[y] &= \int_{\alpha_\varepsilon}^{\beta_\varepsilon} \mathcal{L}(X,Y,Y')\mathrm{d}X - \int_\alpha^\beta \mathcal{L}(x,y,y')\mathrm{d}x \\ &= \int_{\alpha+\varepsilon X_0}^{\beta+\varepsilon X_1} \mathcal{L}(X,Y,Y')\mathrm{d}X - \int_\alpha^\beta \mathcal{L}(x,y,y')\mathrm{d}x = 0 \end{aligned} \tag{4.3}$$

By Theorem 3.1, we know that Y is extremal, and will also satisfy the same Euler-Lagrange equation as y due to the invariance of the lagrangian. Equation 4.3 is precisely a free-endpoint variation problem as discussed in Section 2.2.3, and as both Y and y satisfy the Euler-Lagrange equations via extremality, the only condition that remains is the Free-endpoint condition (Equation 2.4):

$$p\Delta Y(x) - H\Delta X(x)\Big|_{x=\alpha}^{x=\beta} = 0$$

Relation 4.1 and 4.2 for sufficiently small ε yield:

$$(\eta p - \zeta H)\Big|_\alpha^\beta = 0$$

As $[\alpha, \beta]$ was an arbitrary sub-interval of $[a, b]$, we have that:

$$(\eta p - \zeta H) = \text{const.} \; \forall x \in [a, b]$$
$$\iff \frac{\mathrm{d}}{\mathrm{d}t}\left[\eta p - \zeta H\right] = 0 \; \forall x \in [a, b]$$

And thus $\eta p - \zeta H$ is a conserved quantity.

□

Noether's Theorem not only clearly outlines the link between symmetry and conservation, but also explicitly provides a formula for the conserved quantity. Treating empirically derived laws as the result of an underlying theory allows for a more axiomatic treatment of mathematical physics and is an excellent tool for formalising intuition in classical mechanics. To see this in action, we must briefly discuss the following principle:

Theorem 4.2 (Hamilton's Principle)**.** *[5][p. 84] The evolution of a Newtonian system* $\mathbf{q}(t)$ *over a time interval* $[a, b]$ *with lagrangian* $\mathcal{L}(t, \mathbf{q}, \dot{\mathbf{q}})$ *is that which extremises the action functional:*

$$I[q] = \int_a^b \mathcal{L}(t, \mathbf{q}, \dot{\mathbf{q}})\mathrm{d}t$$

We will not prove Hamilton's theorem in this section for the sake of brevity, but the general principle is useful: the paths objects in the physical world take will always extremise some functional, and hence will satisfy the Euler-Lagrange equation. This also establishes a link between the physical laws of a system and its lagrangian – the equations governing the evolution of the system are precisely the Euler-Lagrange equations. This allows us to apply Noether's theorem to a wide variety of situations. We will look at two pedagogical examples in mechanics.

Example 4.1 (Conservation of linear momentum)**.** Consider two particles of masses m_1, m_2 with x-coordinates x_1, x_2 respectively undergoing a 1D collision with no further external forces and gravitational attraction neglected. The canonical choice of lagrangian for this interaction is:

$$\mathcal{L}(t, x_1, x_2, \dot{x_1}, \dot{x_2}) = \frac{1}{2}m_1\dot{x_1}^2 + \frac{1}{2}m_2\dot{x_2}^2$$

We consider the linear transformations:

$$T = \theta(t, x_1, x_2, \dot{x_1}, \dot{x_2}; \varepsilon) \quad = t$$
$$X_1 = \psi_1(t, x_1, x_2, \dot{x_1}, \dot{x_2}; \varepsilon) = x_1 + \varepsilon$$
$$X_2 = \psi_2(t, x_1, x_2, \dot{x_1}, \dot{x_2}; \varepsilon) = x_2 + \varepsilon$$

It is routine to verify these are valid one-parameter symmetries for $\mathcal{L}$ and the corresponding action functional. We calculate the infinitesimal generators:

$$\zeta = \left.\frac{\partial\theta}{\partial\varepsilon}\right|_{(t,\ldots,\dot{x_2};0)} = 0$$
$$\eta_1 = \left.\frac{\partial\psi_1}{\partial\varepsilon}\right|_{(t,\ldots,\dot{x_2};0)} = 1$$
$$\eta_2 = \left.\frac{\partial\psi_2}{\partial\varepsilon}\right|_{(t,\ldots,\dot{x_2};0)} = 1$$

The canonical momenta are given by $p_1 = m_1\dot{x_1}, p_2 = m_2\dot{x_2}$, and we need not calculate the Hamiltonian because $\zeta = 0$. By Noether's Theorem:

$$m_1\dot{x_1} + m_2\dot{x_2} = \text{const.}$$

Which is Newton's law for conservation of momentum. The symmetry we started with has a very natural interpretation: we are simply shifting the positions of each particle along the x-axis. Intuitively, we can see that no matter how we translate the particles along the axis, their motion is uniquely determined by their initial velocities, and as a result the total momentum of the system is conserved.

Example 4.2 (Conservation of angular momentum)**.** Consider the set of transformations defined in Example 3.1, along with the same lagrangian. Again calculating infinitesimal generators, we have:

$$\begin{aligned} \zeta &= \left.\frac{\partial\theta}{\partial\varepsilon}\right|_{(t,\mathbf{q};0)} = 0 \\ \eta_1 &= \left.\frac{\partial\psi_1}{\partial\varepsilon}\right|_{(t,\mathbf{q};0)} = q_2 \\ \eta_2 &= \left.\frac{\partial\psi_2}{\partial\varepsilon}\right|_{(t,\mathbf{q};0)} = -q_1 \end{aligned}$$

The canonical momenta are $p_1 = m\dot{q_1}, p_2 = m\dot{q_2}$, so Noether's theorem states:

$$m\dot{q_1}q_2 - m\dot{q_2}q_1 = \text{const.}$$

By fixing the planetary orbit to the q_1q_2-plane, we can calculate the system's angular momentum using the formula $\mathbf{L} = \mathbf{q} \wedge \mathbf{p}$:

$$\mathbf{L} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ q_1 & q_2 & 0 \\ m\dot{q_1} & m\dot{q_2} & 0 \end{vmatrix} = \hat{\imath}(0) + \hat{\jmath}(0) - \hat{k}(m\dot{q_1}q_2 - m\dot{q_2}q_1) = (\text{const.})\hat{k}$$

So $|\mathbf{L}|$ is constant, and thus the total angular momentum of the system is conserved. This result has a particularly striking implication in astronomy: The Earth is gradually slowing down. It is known that the Moon is slowly drifting away from Earth, but this suggests an increase in the angular momentum of the Earth-Moon system – but we have just proven that this is a conserved quantity, and thus the Earth must *lose* angular momentum to compensate, slowing down in the process. The relationship between conservation and symmetry is one of great pedagogical value, richness, and evidently, interplanetary-scale.

5 References

References

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