Generalized Stoke's Theorem

Student ID: u2226319

April 17, 2024

Contents

1 Introduction

1.1 Aims and Motivation

Theorem 1.1. *[3, p 107][Generalized Stoke's Theorem (Stokes-Cartan Theorem)] Take ω to be a $(n-1)$-form with compact support and let it be smooth. Then let ω be on an n-dimensional manifold M, which has a boundary, denoted ∂M (given the induced orientation). Then we have that*

$$\int_M d\omega = \int_{\partial M} \omega.$$

This essay will aim to build up the mathematical tools to fully understand this statement and prove the theorem. Furthermore, many applications will be explored of the theorem. Stoke's Theorem was communicated from Lord Kelvin to George Stokes by letter in 1850 and led to Stokes setting it for the Smith's Prize exam in 1854 [12] (which is why it is named after himself). In 1861, a simple version of the theorem that was published by Hermann Hankel [6] (Generalized

Stoke's Theorem in lower dimensions). In 1945, Élie Cartan formulated the modern version of the theorem (Theorem 1.1) [14].

Last Year, in Modelling 2, it was suggested that the Fundamental Theorem of Calculus 2, (Classical) Stoke's Theorem, Divergence Theorem and Green's Theorem could all be derived from Generalized Stoke's Theorem [2, p. 90]. This was intriguing and through proving the theorem, this essay will intend to show all can be derived from it. Generalized Stoke's Theorem is considered to be one of the most elegant theorems in mathematics and is used in many fields of mathematics including analysis, topology, and geometry. The theorem applies few conditions and is fundamental (generalises for all situations, rather than special cases) so it is a major building block for other theorems.

1.2 The Theorems

Firstly, let's state 4 theorems, which we want to derive from Generalized Stoke's Theorem. In Year 1, we saw these formulae before but, took them for granted as, they were never proved (except that we proved the Fundamental Theorem of Calculus 2 using Riemann integration techniques but, a different proof will be offered here). It's important to notice how they mirror Generalized Stoke's Theorem, in the fact 1 side of the equation has a 1 less dimension that it is integrating over.

Theorem 1.2 (Green's Theorem). *[2, p. 73] Consider an area D which is bounded by a curve C. If we have that this curve is closed, simple and oriented then for any two-variable functions P, Q that have continuous partial derivatives on D, we have that*

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dxdy = \oint_C (Pdx + Qdy).$$

Theorem 1.3 ((Classical) Stoke's Theorem). *[2, p. 76] We consider the vector field $\underline{F}$, which is a function in $\mathbb{R}^3$. Let S be a surface with unit normal $\underline{\hat{n}}$ and boundary curve C oriented positively, then*

$$\iint_S \operatorname{curl} \underline{F} \cdot \underline{\hat{n}} dS = \int_C \underline{F} \cdot d\underline{r}.$$

Here, $\operatorname{curl} \underline{F} = \nabla \times \underline{F}$ (where ∇ is the divergence operator) and this is the classical version of Generalized Stoke's Theorem published by Hankel.

Theorem 1.4 (Divergence Theorem). *[2, p. 85] We let $\underline{\hat{n}}$ be a unit normal vector to a surface S and additionally, we require that this vector points outwards. Take $\underline{F}$ to be a vector field (again in 3 variables). Let V be a finite volume in $\mathbb{R}^3$ and S is its (closed) surface. Furthermore if $\underline{F}$ was differentiable then:*

$$\iiint_V \nabla \cdot \underline{F} dV = \iint_S \underline{F} \cdot \underline{\hat{n}} dS.$$

Theorem 1.5 (Fundamental Theorem of Calculus 2). *[1, p. 80] Take F to be a real-valued function on $[a, b]$, which we know that it is continuous on $[a, b]$ and differentiable on (a, b). We let $F' = f$ and we see that*

$$\int_a^b f(t)dt = F(b) - F(a),$$

which only holds if we have that f is integrable on $[a, b]$.

Now, before we start trying to understand Generalized Stoke's Theorem, we will see how Green's Theorem is a simpler case of (Classical) Stoke's Theorem therefore, eliminating 1 theorem from the work to do later.

Proposition 1.6. *Green's Theorem can be derived from (Classical) Stoke's Theorem*

Proof. Let $\underline{F} = (P, Q, 0)$, where $P = P(x, y)$ and $Q = Q(x, y)$ and the curve C has a surface of D such that:

$$\nabla \times \underline{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ P & Q & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ Q_x - P_y \end{pmatrix}.$$

Since, this surface is reduced to 2 dimensions (z component is 0), then a unit normal vector is $\underline{\hat{n}} = \left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right)$.

(1) The Left Hand Side of Stoke's Theorem:

$$\iint_D \left(\begin{smallmatrix} 0 \\ 0 \\ Q_x - P_y \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right) dxdy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dxdy.$$

(2) The Right Hand Side of Stoke's Theorem:

$$\int_C \left(\begin{smallmatrix} P \\ Q \\ 0 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} dx \\ dy \\ dz \end{smallmatrix}\right) = \int_C (Pdx + Qdy).$$

The right and left side now correspond to the result which we wanted to prove, therefore in the verification proof that will come in Chapter 6, showing (Classical) Stoke's Theorem is enough to get Green's Theorem. □

2 Building up tools

This section is very important for understanding differential forms as tensors are needed to understand wedge products, which are in turn needed to understand differential forms. This section is presented in a different way that tensors are presented in Multi-Linear Algebra (MA266), with more proofs and different intuition. After tensors and wedge products are understood, we can define differential forms and exterior derivatives.

2.1 Tensors

Definition 2.1. A *multi-linear function* is a function in which each separate variable is linear.

Example 2.1. Assume f is a multi-linear function, consider $f(3x + 2y, y - z, z)$. We can use the linearity of each separate variable to reduce this expression to $3f(x, y - z, z) + 2f(y, y - z, z)$ and again to get $3f(x, y, z) - 3f(x, z, z) + 2f(y, y, z) - 2f(y, z, z)$.

Definition 2.2. [5, p. 153] Take Z to be a vector space then for $k \in \mathbb{N}$, then we define the k-tensor of Z to be a function on Z^k (this is the Cartesian product), which is real-valued and multi-linear. We also require that Z is finite dimensional. For this, the *rank* of the tensor is k. Since Z is a vector space we can assume linearity and scalar multiplication properties of tensors.

Definition 2.3. $T^m(Z)$ is the set of rank m tensors of Z

Definition 2.4. [11, p. 75] Take Z from above and let A be a rank f tensor and B be a rank g tensor. *The tensor product* is defined by $A \otimes B \in T^{f+g}(Z)$ such that:

$$(A \otimes B)(z_1, \ldots, z_f, z_{f+1}, \ldots, z_{f+g}) = A(z_1, \ldots, z_f) \cdot B(z_{f+1}, \ldots, z_{f+g})$$

.

Proposition 2.1. *The tensor product is distributive,* $A \otimes (B_1 + B_2) = A \otimes B_1 + A \otimes B_2$.

Proof. In this proof we just going to do a routine expansion of terms to get the result we want. Let A have rank f and, B_1 and B_2 have rank g. Then we see that:

$$\begin{aligned}(A \otimes (B_1 + B_2))(z_1, \ldots, z_{f+g}) &= A(z_1, \ldots, z_f) \cdot (B_1 + B_2)(z_{f+1}, \ldots, z_{f+g}) \\ &= A(z_1, \ldots, z_f) \cdot (B_1(z_{f+1}, \ldots, z_{f+g}) + B_2(z_{f+1}, \ldots, z_{f+g})) \\ &= (A \otimes B_1 + A \otimes B_2)(z_1, \ldots, z_f, \ldots, z_{f+g}).\end{aligned}$$

□

Proposition 2.2. *The tensor product is associative.*

Proof. Likewise, in this proof we just going to do a routine expansion of terms to get the result we want. Let A have rank f, let B have rank g and C have rank h. Then we see that:

$$\begin{aligned}((A \otimes B) \otimes C)(z_1, \ldots, z_{f+g+h}) &= (A(z_1, \ldots, z_f) \cdot B(z_{f+1}, \ldots, z_{f+g})) \cdot C(z_{f+g+1}, \ldots, z_{f+g+h}) \\ &= A(z_1, \ldots, z_f) \cdot (B(z_{f+1}, \ldots, z_{f+g}) \cdot C(z_{f+g+1}, \ldots, z_{f+g+h})) \\ &= (A \otimes (B \otimes C))(z_1, \ldots, z_{f+g+h}).\end{aligned}$$

□

Definition 2.5. [11, p. 78] Tensors are *alternating* if in the domain (the Cartesian product), when we interchange 2 z_i and z_j, where $i \neq j$, then the output of the tensor is the negative value of what it was. [8, p. 314] We call a tensor that is equal, when 2 components are interchanged *symmetric.*

Definition 2.6. [5, p. 155] Let $X \in T^p(Z)$ then:

$$(AltX)(z_1, \ldots, z_p) = \frac{1}{p!} \sum_{\sigma \in S_p} (\operatorname{sgn} \sigma) X(z_{\sigma(1)}, \ldots, z_{\sigma(p)})$$

and it is an alternating tensor, where S_p is the symmetric group defined in Algebra 1.

Proposition 2.3. *$AltX$ is an alternating tensor such that $[AltX]^\phi = \operatorname{sgn}(\phi) AltX$, where $[AltX]^\phi$ is $AltX$ with permutation ϕ applied to it.*

Proof. Let $X \in T^p(Z)$. In this proof we just apply a random permutation to $(AltX)$ and then consider the composition of the 2 permutations (the other being that used in the definition of $(AltX)$) as 1 permutation to derive the result. Firstly, we can see that

$$[(AltX)(z_1, \ldots, z_p)]^\phi = \frac{1}{p!} \sum_{\sigma \in S_p} (\operatorname{sgn} \sigma) X(z_{\phi(\sigma(1))}, \ldots, z_{\phi(\sigma(p))}),$$

and we know $\operatorname{sgn}(\phi \circ \sigma) = \operatorname{sgn}(\phi) \operatorname{sgn}(\sigma)$. Therefore, can insert $\operatorname{sgn}(\phi)$ twice as $\operatorname{sgn}(\phi)^2 = 1$ and set $\tau = \phi \circ \sigma$, with $\tau \in S_p$. We can now show

$$\begin{aligned}[(AltX)(z_1, \ldots, z_p)]^\phi &= \frac{1}{p!} \operatorname{sgn}(\phi) \sum_{\sigma \in S_p} (\operatorname{sgn}(\phi \circ \sigma)) X(z_{\phi(\sigma(1))}, \ldots, z_{\phi(\sigma(p))}) \\ &= \frac{1}{p!} \operatorname{sgn}(\phi) \sum_{\sigma \in S_p} (\operatorname{sgn} \tau) X(z_{\tau(1)}, \ldots, z_{\tau(p)}) \\ &= \operatorname{sgn}(\phi) AltX(z_1, \ldots, z_p),\end{aligned}$$

as τ becomes a dummy-variable in the series. Now if ϕ is a transposition (swapping v_i and v_j), then $\operatorname{sgn}(\phi) = -1$ due to a transposition being odd, therefore showing $AltX$ is an alternating tensor. □

2.2 Wedge Products

We now define wedge product from our understanding of tensors.

Definition 2.7. [8, p. 355] Let $A \in T^f(Z)$ and $B \in T^g(Z)$ then the *wedge product* defined by:

$$A \wedge B = \frac{(f+g)!}{f!g!} Alt(A \otimes B).$$

Proposition 2.4. *Let A,B and C be tensors. Wedge products have associativity property $(A \wedge B) \wedge C = A \wedge (B \wedge C)$ and anti-commutativity property $A \wedge B = (-1)^{fg} B \wedge A$ (if A was chosen to have rank f and B rank g).*

The above proposition is easy to prove as they are just routine verifications and very similar to the proofs provided for tensor products. The key step to the proof of the anti-commutativity property is that we already know that $AltX$ is an alternating tensor. These are left as an exercise for the reader. We use these properties a lot and they are crucial to understanding differential forms. The combination of these 2 properties will become powerful in the proof of Generalized Stoke's Theorem.

2.3 Forms

Definition 2.8. [8, p. 351] Let $k \in \mathbb{N}$ and consider positive integers i_j for $j \in \{1, \dots, p\}$. A multi-index of length p is a p-tuple $I = (i_1, \dots, i_p)$. If I is a multi-index and $\sigma \in S_p$, then $I_\sigma = (i_{\sigma(1)}, \dots, i_{\sigma(p)})$.

Definition 2.9. [4, p 157] A *differential p-form* is just the sum:

$$\omega = \sum_{i_1 \cdots i_p} \omega_{i_1,\dots,i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}.$$

We have that $\omega_{i_1 \cdots i_p} \in C^\infty(U)$ (smooth functions from U) and indices $1 \leq i_1 < \cdots < i_p \leq m$. The differential form is on an open subset of $\mathbb{R}^m$ here. With the former notation, we can write this as $w = \sum_I w_I dx^I$ with $I = (i_1, \dots, i_p)$ being an index set and $dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_p}$ and $w_{i_1 \cdots i_k} = w_I$. Here, dx^i means the ith coordinate, not x to the power of i.

Definition 2.10. [4, p 158] Let $\Omega^p(U)$ be the vector space (with addition and scalar multiplication) of p-forms on U. Define the *exterior derivative* to be a map $d : \Omega^p(U) \to \Omega^{p+1}(U)$ such that:

$$d\left(\sum_I w_I dx^I\right) = \sum_{i=1}^{p} \sum_I \frac{\delta w_I}{\delta x^i} dx^i \wedge dx^I.$$

Proposition 2.5. *The exterior derivative has the property $d(a+b) = da + db$, for every $a, b \in \Omega^m$.*

Proof. Let $a = \sum_I a_I dx^I$ and $b = \sum_I b_I dx^I$, then we see that:

$$\begin{aligned}
d(a+b) &= d\left(\sum_I (a_I + b_I) dx^I\right) \\
&= \left(\sum_I d(a_I + b_I) \wedge dx^I\right) \\
&= \left(\sum_I (da_I + db_I) \wedge dx^I\right) \\
&= \left(\sum_I da_I \wedge dx^I + db_I \wedge dx^I\right) \\
&= da + db
\end{aligned}$$

□

Proposition 2.6. *[4, p 158] The exterior derivatives has the property $d \circ d = 0$ hence, $dd\omega = 0$ for every $\omega \in \Omega^m$.*

Proof. If we apply the definition twice then:

$$dd\omega = \sum_{j=1}^{m} \sum_{i=1}^{m} \sum_{I} \frac{\partial^2 w_I}{\partial x^j \partial x^i} dx^j \wedge dx^i \wedge dx^I.$$

Since, $dx^i \wedge dx^j = -dx^j \wedge dx^i$ and we know mixed derivative are equal so the coefficients in front of $dx^i \wedge dx^j$ and $dx^j \wedge dx^i$ are equal. Therefore, the terms cancel out (due to 1 being the negative of the other) leading to 0. □

The proofs above were easy enough as they were just routine expansions but the power we have unlocked from these propositions is immense. These properties will be used time and time again during the proof of Generalized Stoke's Theorem.

3 Manifolds

Manifolds are one of the most useful constructs in all of mathematics. The use of a manifold is to generalise surfaces and curves into higher dimensions. We think of a manifold as a topological space that locally can be seen as a Euclidean space near each point and this gives us tools to use on them that we already have built up (a more formal definition is offered below). This property of being locally Euclidean can be described as every point having a neighbourhood (this will be defined as a chart) which is homeomorphic to an open subset of $\mathbb{R}^n$. These coordinate charts can be worked with to allow us to get properties over manifolds which we want such as differentiation, tangent spaces and differential forms being able to be built up. This section will build up the tools to understand these concepts and will be crucial in understanding Generalized Stoke's Theorem. In addition, this essay will assume knowledge from Norms, Metrics and Topologies (MA260) such as: a topology, support of a function (denoted *supp*), a neighbourhood, covers, connectedness, homeomorphisms, compact sets, closure and a boundary.

3.1 Introduction to Manifolds

Definition 3.1. A *p-dimensional manifold (p-manifold)* is a topological space with the property that each point has a neighbourhood that is homeomorphic to an open subset of $\mathbb{R}^p$.

Example 3.1. 1-manifolds include lines and circles and 2-manifolds include planes, spheres, and toruses. Something interesting is that our intuition on 3-manifolds is limited to just a theoretical level where, we can use them in equations. What this means is that, we are not sure what 3-manifolds actually look like for example, some people theorise it could be the shape of the universe.

Example 3.2. The simplest type of manifold is a topological manifold. We say that a topological space with a basis (that is made up of a countable set of subsets) is *second-countable* [9]. A manifold that locally resembles a p-dimensional Euclidean space is topological (denoted M^p) if it is both Hausdorff and second-countable [10, p. 35]. Other types of manifolds include differentiable manifolds and smooth manifolds.

3.2 Definitions

This section is dense containing lots of important definitions used throughout the essay. We build on the notion of smooth maps in the next subsection when looking at smooth manifolds. In Multi-variable Analysis (MA263), we defined a smooth map so we will quickly define it, then go onto diffeomorphisms from it.

Definition 3.2. [4, p. 19] Consider a function $G : A \to B$, where it is *smooth* if we can infinitely differentiate it. This for A an open subset of $\mathbb{R}^p$ and B an open subset of $\mathbb{R}^q$. This means we are allowed to differentiate to get the n^{th} derivative for any $n \in \mathbb{N}$. The set all smooth functions from A to B is denoted $C^\infty(A, B)$

Definition 3.3. [4, p. 19] $G : A \to B$ is smooth and it is a *diffeomorphism* if it is bijective (this implies that is invertible) and $G^{-1} : B \to A$ is smooth (here $p = q$).

Example 3.3. A simple example of a diffeomorphism is $f(x) = x$, which is infinitely differentiable and self-inverting. Another example is $f(x) = x^3 + x$, since $f'(x) = 3x^2 + 1$, f is strictly increasing and therefore, it is injective. Furthermore, as $x \to \pm\infty$, $f(x) \to \pm\infty$ so it is surjective hence, it is a bijection. Clearly, this is infinitely differentiable too. Using the inverse function theorem, we can see this is a diffeomorphism.

Now we want a way to express the fact that at each point there exists a neighbourhood that is homeomorphic to an open subset of a Euclidean space. To do this we create a pair consisting of a map and the neighbourhood.

Definition 3.4. [4, p. 20] A *p-dimensional coordinate chart* is a collection of a chart domain (Q) and a coordinate map (ϕ). It is represented as (Q, ϕ) and on the set M, the coordinate map is defined by $\phi : Q \to \mathbb{R}^p$ and it is a bijection if restricted from Q to $\phi(Q)$.

Definition 3.5. [4, p. 20] Two charts are *compatible* if we can construct a specific diffeomorphism between the coordinate maps. Take (Q, ϕ) and (R, σ) that are compatible, then $\sigma \circ \phi^{-1} : \phi(Q \cap R) \to \sigma(Q \cap R)$ is diffeomorphism, where $\phi(Q \cap R)$ and $\sigma(Q \cap R)$ were open subsets of their respective chart domains. We call these maps the transition maps. Furthermore, if $Q \cap R = \varnothing$ then it is always compatible.

Charts on their own are not that useful, we want to make a notion that relates the charts on a manifold together.

Definition 3.6. [4, p. 22] Take M to be a set. An *atlas* that acts on M is a collection of charts that satisfy 2 properties: the chart domains cover M and each chart in the atlas is compatible to every other chart in it. We denote an atlas by $A = \{(Q_\alpha, \phi_\alpha)\}$, where A is an atlas on M.

3.3 Smooth Manifolds

The importance of atlases is that we can now consider a structure that covers the manifold and work with it to derive properties we wish the manifold to have.

Definition 3.7. [3, p. 5] *A smooth atlas* is an atlas, where the notion of a function being smooth on a manifold is known. A function $g : M \to \mathbb{R}$ is smooth if $g \circ \phi^{-1}$ is a smooth real-valued map for all charts (the input) inside our atlas. 2 smooth atlases here are compatible if the transition map, suggested in Definition 3.5, is smooth for all charts in the atlas.

A smooth manifold is a topological manifold in which differentiation is possible. Every smooth manifold is a topological manifold but the converse isn't true.

Definition 3.8. [3, p. 5] A smooth manifold is a manifold with a smooth structure, which is a structure on M defined by the equivalence classes of smooth atlases that act on M. The smooth structure ensures that for 2 atlases representing these equivalence classes, their union is also a smooth atlas. This means it has a C^∞-differentiable structure in which we can infinitely differentiate due to the smoothness.

This definition is too formal and a bit confusing, the main take away here is that on a smooth manifold, we have ensured that there is differentiation at each point.

Definition 3.9. [3, p. 105] To define the boundary of this smooth manifold we consider a chart where the coordinate map instead maps to $\mathbb{H}^p = \{(x_1, \ldots, x_p) : x_p \geq 0\}$, where the boundary is the smooth atlas of the chart domains and coordinate maps. Furthermore, we require that the transition maps are diffeomorphisms and these maps are between open sets of $\mathbb{H}^p$ as the domain is $\mathbb{H}^p$.

Example 3.4. The most basic example of a smooth manifold is $\mathbb{R}^p$. Another includes, the $2p$-dimensional vector space $\mathbb{C}^p$, the complex p-space.

Now, we have a notion of what manifolds are and the smoothness property they possess. Furthermore, we know what the boundary of a manifold is too now. To understand Generalized Stoke's Theorem we need to combine Chapter 2 and 3 in order to get a notion of integration using these concepts.

4 Tools for Integration

4.1 Orientation

This is very important section as orientation must be considered when integrating over manifolds. Orientation here is analogous to orientation of bases defined in Algebra 2, where there were 2 orientations (used for parallelepipeds). This section is brief, as for Generalized Stoke's Theorem an appreciation of orientation and just being considerate is enough. If we were to formally define orientation we would build upon tangent spaces learnt about in Multi-variable Analysis (MA263) and assign an orientation to each point in the tangent space.

Definition 4.1. [13, p 240] A manifold is *orientable* if it has an orientation. There are only 2 possible orientations of a manifold that is both orientated and connected.

4.2 Integrating Forms

The final concept we must explore before we tackle the proof of Generalized Stoke's Theorem. We write a slightly easier way to visualise forms and then introduce integration over it. We simplify our definition of a differential form then we integrate over open sets, charts and then manifolds.

Definition 4.2. [4, p 193] Let ω be a form that has the highest degree possible (in $\mathbb{R}^m$ this is m) that is on U which is an open subset of $\mathbb{R}^m$. The differential can be written as:

$$\omega = f dx^1 \wedge \cdots \wedge dx^m,$$

where $f \in C^\infty(U)$.

This definition was powerful as in further calculations we going to use this definition as we are going to be working with top degree forms (this means that the degree of the form is the same as the dimension of the space we are integrating over).

Definition 4.3. [4, p 193] If supp(f) (this is a short hand for the support of f, the region in it isn't 0) is compact (bounded and closed), then we define *the integral over manifolds* of forms as:

$$\int_U w = \int_{\mathbb{R}^m} f(x^1, \dots, x^m) dx^1 \cdots dx^m,$$

where U is defined in Definition 4.2.

This definition considered the whole of $\mathbb{R}^m$ as integration over $\mathbb{R}^m \backslash U$ is 0 due to the compact support, therefore adding 0 onto the integral we want does not matter.

Definition 4.4. [13, p 265] Take a p-form ω which is on U, such that it has a compact support, where this U is defined to be the chart domain of a chart (U, ϕ) and this is from an atlas of M (a manifold with dimension p). If we take the coordinate map ϕ as defined in the chart definition but restrict the domain to $\phi(U)$, then we gain a bijection and this is also a diffeomorphism. Now we can see that we have defined a new p-form, which is $(\phi^{-1})\omega$ and this has compact support on $\phi(U)$. One can view this is as us constructing a differential form that acts on the image of U under the coordinate map. Now we can define integration over a chart domain of a differential form as:

$$\int_{\phi(U)} (\phi^{-1})\omega = \int_U \omega$$

Proposition 4.1. *[13, p 265] Definition* 4.4 *is well-defined.*

Proof. To prove the definition is well-defined we must get 2 different charts (with the same chart domain but different coordinate map) and show the result is the same. Let (U, σ) be a chart in the atlas with the same U, then $\phi \circ \sigma^{-1} : \sigma(U) \to \phi(U)$ is a diffeomorphism. Then we see that:

$$\int_{\phi(U)} (\phi^{-1})\omega = \int_{\sigma(U)} (\phi \circ \sigma^{-1})(\phi^{-1})\omega = \int_{\sigma(U)} (\sigma^{-1})\omega,$$

the integration of ω over a chart is well-defined and independent of choice of coordinate chart over U. ☐

We now define partition of unities, which allow us to build a global object on manifolds, rather than local ones.

Definition 4.5. [3, p. 21] A *partition of unity* on a smooth manifold M is the set of smooth functions $\{\rho_i\}$, such that the functions act on M and $i \in I$ for the multi-index I. We also require that these functions satisfy 3 conditions: they must all be greater than 0, they sum to 1 and the collection of the functions' supports are locally finite. The locally finite property means that if we take a point in M and consider a neighbourhood around it, then it only meets the support of a finite amount of these functions. This further implies in this neighbourhood the sum of the functions is indeed finite.

Definition 4.6. [3, p. 21] As in the previous definition, we take $\{\rho_i\}$ as our partition of unity satisfying those conditions. There is an open cover of M, which is union of some open sets V_j. The partition of unity is subordinate to the open cover if for all $i \in I$ there is some j such that the support of the function ρ_i is contained within V_j.

Now we want to use partitions of unity in order to integrate over manifolds as we now have a global structure, [13, p 265]. For a partition of unity $\{\rho_i\}$ subordinate to the open cover $\{V_i\}$, we can give ω compact support and by the definition of partition of unity, it is locally finite (there are finitely many $\rho_i\omega$ that are non-zero). Then we know that $\omega = \sum_i \rho_i\omega$ is a finite sum (by definition) and we know that supp($\rho_i\omega$) is compact (this is due to it being finite and ω had compact support). We can now work with the n-form $\rho_i\omega$ that has compact support in the chart V_i. The integration is well-defined (any partition of unity yields the same result) over a manifold using these ideas and given below.

Definition 4.7. [13, p 265] We can define *the integration of ω over manifold M* to be the finite sum:

$$\int_M \omega = \sum_i \int_{V_i} \rho_i \omega.$$

Lemma 4.2. *[13, p 266] Integrating a form over the same manifold but now given opposite orientation to before, will cause the integral to be the negative of what it was.*

5 The Proof

Now, with all the tools built up in previous sections, the proof of Generalized Stoke's Theorem (Theorem 1.1) will be presented. The proof shows that Generalized Stoke's Theorem works if it works for 2 special cases, which are then explored. This proof is based on the proofs given in Loring W. Tu's, "An Introduction to Manifolds" [13, p 269–270] and John M. Lee's, "Introduction to Smooth Manifolds" [8, p. 411–413]. The general claim is from Tu's book and the 2 specific cases are from Lee's book. We follow the same steps for specific parts of each proof but the proof offered here will give explanations for every single step and tie the proofs together into a concise proof of Generalized Stoke's Theorem. Both proofs in the books, left out explanation and a lot was assumed of the reader so here it will be explained fully. In addition, the reader can draw parallels to the proof of Green's Theorem in Multi-variable Analysis (MA263) as a similar approach to the proof is done here.

Proof. Consider the atlas $\{(U_\alpha, \phi_\alpha)\}$, where U_α is diffeomorphic to either $\mathbb{H}^n$ or $\mathbb{R}^n$. We take the partition of unity $\{\rho_\alpha\}$ such that it satisifies the properties in Defintion 4.5, furthermore we require that it is subordinate to $\{U_\alpha\}$. We take a $(n-1)$-form $\rho_\alpha \omega$ such that $supp(\rho_\alpha \omega) \subset U_\alpha$. We can claim that if the theorem holds for just the cases $M = \mathbb{H}^n$ and $M = \mathbb{R}^n$, then it holds for all charts in the atlas, which are diffeomorphic to either one of them. Furthermore, we know that $(\partial M) \cap U_\alpha = (\partial U_\alpha)$. Then we have that:

$$\begin{aligned}\int_{\partial M} w &= \int_{\partial M} \sum_\alpha \rho_\alpha \omega = \sum_\alpha \int_{\partial M} \rho_\alpha \omega \\ &= \sum_\alpha \int_{\partial U_\alpha} \rho_\alpha \omega = \sum_\alpha \int_{U_\alpha} d(\rho_\alpha \omega) \\ &= \sum_\alpha \int_M d(\rho_\alpha \omega) = \int_M d\left(\sum_\alpha \rho_\alpha \omega\right) \\ &= \int_M dw.\end{aligned}$$

Most of the things used in my explanation here are from Definition 4.5 and 4.6 for the partition of unity. This integration follows from the fact that firstly, the sum of partitions of unity is equal to 1 so can be inserted into the integral. Then we use the fact that this sum is finite so, we can freely swap the sum and integral. Then we reduce what we are integrating over as supp$(\rho_\alpha \omega)$ is contained in U_α as, $\rho_\alpha \omega$ has compact support in it. Then Generalized Stoke's Theorem was assumed to hold so we can use it and we also use the fact that supp$(d(\rho_\alpha)) \subset U_\alpha$ to get to integrating over a manifold. Finally, we again use the fact that the sum is finite to get the result we desired. Therefore, it suffices to prove the theorem for $\mathbb{H}^n$ and $\mathbb{R}^n$.

Suppose M is the upper half-space $\mathbb{H}^n$. We can chose some real number $R > 0$ such that $supp(\omega) \subset B = [-R, R]^{n-1} \times [0, R]$ due to the fact we defined ω to have compact support. We know that outside this region ω is 0 (we can only do this as the support is compact). Now, using Definition 4.2 we are going to write ω in terms of a smooth function and wedge products such that:

$$\omega = \sum_{i=1}^{n} f_i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots dx^n,$$

and we have a hat above dx^i as it is omitted. This is because we are just using the usual form formula but we have a $(n-1)$-form on a n-manifold, so we can eliminate 1 exterior derivative (but we don't know which we removed so it's a sum). Then we can see that:

$$\begin{aligned}
d\omega &= \sum_{i=1}^{n} df_i \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots dx^n \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial f_i}{\partial x^j} dx^j \wedge dx^1 \wedge \cdots \widehat{dx^i} \wedge \cdots \wedge dx^n \\
&= \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial f_i}{\partial x^i} dx^1 \wedge \cdots \wedge dx^n.
\end{aligned}$$

Firstly we just apply a derivative to each side. Then we substitute in the definition of an exterior derivative in place of df_i, producing a double sum. However, now from Proposition 2.6, we know the a exterior derivative applied to itself is 0, hence every term of the sum over j is 0 except for when $j = i$ as dx^i was chosen to be omitted. Finally the $(-1)^{i-1}$ term appears when reordering dx^i so the exterior derivatives are in numerical order. This is because wedge products obey Proposition 2.4 so if dx^i is moved an odd number of times to be in place there is a negative sign. Now we consider the integral of $d\omega$ over the half-space:

$$\begin{aligned}
\int_{\mathbb{H}^n} d\omega &= \sum_{i=1}^{n} (-1)^{i-1} \int_B \frac{\partial f_i}{\partial x^i} dx^1 \wedge \cdots \wedge dx^i \\
&= \sum_{i=1}^{n} (-1)^{i-1} \int_0^R \int_{-R}^{R} \cdots \int_{-R}^{R} \frac{\partial f_i}{\partial x^i} dx^1 \cdots dx^n.
\end{aligned}$$

We assume this obeys nice properties meaning summation and integration can be swapped. We can rearrange the order of integration and substitute in what B was, using the Cartesian product we defined and can get rid of the wedge products, seen in Definition 4.3. The reason we are integrating over B is that the $\operatorname{supp} \omega$ is contained in it (the rest of $\mathbb{H}^n$ when integrated over would be 0). Now we consider for x^i terms such that $i \neq n$ and we can apply the Fundamental Theorem Calculus here to get:

$$\sum_{i=1}^{n-1} (-1)^{i-1} \int_0^R \int_{-R}^{R} \cdots \int_{-R}^{R} \frac{\partial f_i}{\partial x^i} dx^1 \cdots dx^n$$

$$= \sum_{i=1}^{n-1} (-1)^{i-1} \int_0^R \int_{-R}^{R} \cdots \int_{-R}^{R} \frac{\partial f_i}{\partial x^i} dx^i dx^1 \cdots \widehat{dx^i} \cdots dx^n$$

$$= \sum_{i=1}^{n-1} (-1)^{i-1} \int_0^R \int_{R}^{-R} \cdots \int_{R}^{-R} f_i \Big|_{x^i=-R}^{x^i=R} dx^1 \cdots \widehat{dx^i} \cdots dx^n = 0.$$

The sum decreased from summing to n to summing to $n-1$ as we can considering all the values of $i \neq n$. We are going to integrate over dx^i first, therefore in the product of other exterior derivatives we put $\widehat{dx^i}$ to show it is now omitted there. We can use the Fundamental Theorem Calculus as ω is continuous and differentiable and the partial derivative is integrable. We then get bounds for $f_i(x)$, but we can chose R large enough so it lies outside the support meaning ω

here evaluates to 0 when $x^i = \pm R$. Therefore, the result of the integration here is just 0. Now we have to check whether for $i = n$ it is 0 or something else. We see that:

$$\begin{aligned}\int_{\mathbb{H}^n} d\omega &= (-1)^{i-1} \int_{-R}^{R} \cdots \int_{-R}^{R} \int_{0}^{R} \frac{\partial f_n}{\partial x^n}(x) dx^n dx^1 \cdots dx^{n-1} \\ &= (-1)^{i-1} \int_{-R}^{R} \cdots \int_{-R}^{R} f_n(x)\Big|_{x^n=0}^{x^n=R} dx^1 \cdots dx^{n-1} \\ &= (-1)^{i-1} \int_{-R}^{R} \cdots \int_{-R}^{R} f_n(x^1, \ldots, x^{n-1}, 0) dx^1 \cdots dx^{n-1}.\end{aligned}$$

We omit $\widehat{dx^n}$ as it is obvious we removed it. We try the same integral again, remembering Fubini's Theorem (Analysis 3) can be applied to do the integral over dx^n first. Here we again use the Fundamental Theorem of Calculus and choose R large enough such that $f_n = 0$ when $x^n = R$. The difference here is one of the limits is 0 so we get a value of f_n here, hence the result is non-zero. Therefore, the results calculated for $i = n$ is what the left side of Generalized Stoke's Theorem is equal to here.

The case where $M = \mathbb{R}^n$, $supp(\omega) \subset B = [-R, R]^n$. The same computation above gives us that all terms vanish (including when $i = n$ as now the lower boundary of the integral isn't 0, it is $-R$) so, the left side of Generalized Stoke's Theorem is 0.

Now, we need to look at the other side of the Generalized Stoke's Theorem, to find that:

$$\int_{\partial\mathbb{H}^n} \omega = \sum_i \int_{B\cap\partial\mathbb{H}^n} f_i(x^1, \ldots, x^{n-1}, 0) dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots dx^n.$$

We know that on $\partial\mathbb{H}^n$ that x^n vanishes (as the boundary is the same of as $\mathbb{H}^n$ but the n^{th} coordinate is 0) and it's derivative here is also zero. From earlier, we know for $i \neq n$ it goes to 0 so consider $i = n$, leading to:

$$\int_{\partial\mathbb{H}^n} \omega = \int_{B\cap\partial\mathbb{H}^n} f_i(x^1, \ldots, x^{n-1}, 0) dx^1 \wedge \cdots \wedge dx^{n-1}.$$

The coordinates $(x_1, \ldots, x_{n-1})$ are positively orientated on $\partial\mathbb{H}^n$ when n is even and negatively orientated when n is odd. We can assign orientations to ± 1 as on connected manifolds there are only 2 orientations. The expression becomes:

$$\int_{\partial\mathbb{H}^n} \omega = (-1)^{n-1} \int_{-R}^{R} \cdots \int_{-R}^{R} f_i(x^1, \ldots, x^{n-1}, 0) dx^1 \cdots dx^{n-1}.$$

Therefore, $\int_{\mathbb{H}^n} dw = \int_{\partial\mathbb{H}^n} \omega$ so it is true for M as the upper-half space.

For $M = \mathbb{R}^n$ it has a empty-boundary (by definition) so the right side of Generalized Stoke's Theorem is 0 too, therefore both sides of the equation are equal. Hence, it is also true for M as $\mathbb{R}^n$. □

6 Linking back to the motivation

6.1 Deriving the Theorems

Theorem 6.1. *All the theorems listed in Chapter 1 can be derived from Generalized Stoke's Theorem.*

Proof. This proof uses ideas from Feldman, Rechnitzer and Yeager's "CLP-4 Vector Calcus" [7, p 256-260].

Firstly, lets establish some basic properties about integrating with differential forms and these properties were proved in Modelling 2 [2] (without the notion of a form).

- [2, p 71] Let $\omega = \underline{F}(\underline{r}) = F_1 dx + F_2 dy + F_3 dz$ be a 1-form. Let C be a curve parametrised by $\underline{r}(t)$ then we have that:
$$\int_C \omega = \int_C \underline{F} \cdot d\underline{r}. \tag{1}$$
- [2, p 83] Let $\omega = F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy$ be a 2-form. Let S be an orientated surface that is parametrised by $\underline{r}(t) = (x(u,v), y(u,v), z(u,v))$ such that $\hat{\underline{n}} dS = +\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} dudv$ (correct orientation). Then we see that:
$$\int_S \omega = \iint_S \underline{F} \cdot \hat{\underline{n}} dS. \tag{2}$$
- This equation wasn't in Modelling 2 but it is an obvious application of what this essay has discussed about forms. Let $\omega = F dx \wedge dy \wedge dz$ be a 3-form and let V be a volume in $\mathbb{R}^3$, then:
$$\int_V \omega = \iiint_V F dxdydz. \tag{3}$$

Firstly, we will prove the Fundamental Theorem of Calculus 2. Here, we have that $M = [a, b]$ (a 1-dimensional manifold) with the 0-form ω on M. Clearly the boundary is just 2 points, $\partial M = \{a, b\}$. A 0-form is a just a function $F(x)$. We have that $F' = f$ therefore, $dF = f dx$. Also, f is Riemann integrable due to Generalised Stoke's Theorem. Now, lets plug into Generalized Stoke's Theorem:
$$\int_a^b f(x)dx = \int_{\{a,b\}} F(x).$$
The manifold [a,b] is orientable and it is connected (learnt in Norms, Metrics and Topologies) therefore there can only be 2 orientations are on it. Since, the manifold $[a, b]$ is orientated going left to right, we use the fact that orientation is induced onto the boundary to conclude that a has negative orientation of -1 and b has positive orientation of $+1$. Therefore, $\int_{\{a,b\}} F(x) = F(b) - F(a)$.

Secondly, we will prove (Classical) Stoke's Theorem. Here, we have that the manifold, a 2-dimensional surface, S and the boundary is a positively oriented curve, C, which has dimension of 1. A 1-form, ω can be expressed as $F_1 dx + F_2 dy + F_3 dz$. We can now calculate the exterior derivative:

$$\begin{aligned}
d\omega &= d(F_1 dx + F_2 dy + F_3 dz) \\
&= dF_1 \wedge dx + dF_2 \wedge dy + dF_3 \wedge dz \\
&= \left(\frac{\partial F_1}{\partial x}dx + \frac{\partial F_1}{\partial y}dy + \frac{\partial F_1}{\partial z}dz\right) \wedge dx + \left(\frac{\partial F_2}{\partial x}dx + \frac{\partial F_2}{\partial y}dy + \frac{\partial F_2}{\partial z}dz\right) \wedge dy \\
&+ \left(\frac{\partial F_3}{\partial x}dx + \frac{\partial F_3}{\partial y}dy + \frac{\partial F_3}{\partial z}dz\right) \wedge dz \\
&= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) dy \wedge dz + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) dz \wedge dx + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx \wedge dy \\
&= (\nabla \times \underline{F})_1 dy \wedge dz + (\nabla \times \underline{F})_2 dz \wedge dx + (\nabla \times \underline{F})_3 dx \wedge dy.
\end{aligned}$$

Now, we can use 1 for the right-side of Generalized Stoke's Theorem. We can also now use 2 but with $\underline{F}$ replaced with $\nabla \times \underline{F}$ for the left-side of Generalized Stoke's Theorem. This yields the result:
$$\iint_S \nabla \times \underline{F} \cdot \hat{\underline{n}} = \iint_S \operatorname{curl} F \cdot \hat{\underline{n}} = \int_C \underline{F} \cdot d\underline{r}.$$

Finally, lets verify the Divergence Theorem. Here, we have that the manifold is a 3-dimensional volume, V and the boundary S is a closed surface. A 2-form, ω can be expressed as $F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy$. We can now calculate the exterior derivative:

$$\begin{aligned} d\omega &= dF_1 \wedge dy \wedge dz + dF_1 \wedge dy \wedge dz + dF_1 \wedge dy \wedge dz \\ &= \left(\frac{\partial F_1}{\partial x}dx + \frac{\partial F_1}{\partial y}dy + \frac{\partial F_1}{\partial z}dz\right) \wedge dy \wedge dz + \left(\frac{\partial F_2}{\partial x}dx + \frac{\partial F_2}{\partial y}dy + \frac{\partial F_2}{\partial z}dz\right) \wedge dz \wedge dx \\ &+ \left(\frac{\partial F_3}{\partial x}dx + \frac{\partial F_3}{\partial y}dy + \frac{\partial F_3}{\partial z}dz\right) \wedge dx \wedge dy \\ &= \left(\frac{\partial F_1}{\partial x}dx + \frac{\partial F_2}{\partial y}dy + \frac{\partial F_3}{\partial z}dz\right) dx \wedge dy \wedge dz \\ &= \nabla \cdot \underline{F} dx \wedge dy \wedge dz. \end{aligned}$$

This simplified nicely due to Proposition 2.6. Now, we can use 2 for the right-side of Generalized Stoke's Theorem. We can also now use 3 but with $f = \nabla \cdot \underline{F}$ for the left-side of Generalized Stoke's Theorem. This yields the result:

$$\iiint_V \nabla \cdot \underline{F} = \iint_S \underline{F} \cdot \underline{\hat{n}}.$$

Since, we did a proof of Proposition 1.6 (verifying Green's Theorem) we are done. □

6.2 Conclusion

In conclusion, Generalized Stoke's Theorem is one of the most important theorems in mathematics as it relates different dimensions in such a simple equation. Even though, it took a lot of effort to fully unpack the statement and get to a proof, the benefits of doing this are enormous as we can finally verify theorems we had been taking for granted since Year 1. These theorems have big implications in physics such as Stoke's Theorem being used in electromagnetism for example Maxwell's equations and Divergence Theorem being used for the change of density of fluids.

References

[1] Keith Ball. *Analysis II: MA139.* Unpublished, Warwick University, 1st edition, 2023.

[2] Siri Chongchitnan. *Methods of Mathematical Modelling II.* Unpublished, Warwick University, 1st edition, 2023.

[3] Claudio Gorodski. *Smooth Manifolds.* Switzerland, 1st edition, 2020.

[4] Gal Gross and Eckhard Meinrenken. *Manifolds, Vector Fields, and Differential Forms.* Oxford, 1st edition, 2023.

[5] Vietor Guillemin and Alan Pollaek. *Differential Topology.* Prentice-Hall, Massachusetts Institute of Technology, 1st edition, 1974.

[6] Hermann Hankel. *Zur allgemeinen Theorie der Bewegung der Flüssigkeiten.* Dieterische University, 1st edition, 1861.

[7] Andrew Rechnitzer Joel Feldman and Elyse Yeager. *CLP-4 Vector Calculus.* University of British Columbia, 1st edition, 2023.

[8] John M. Lee. *Introduction to Smooth Manifolds.* New York, 2nd edition, 2003.

[9] Daniel Luckhardt and Urs Schreiber. Second-countable space. https://ncatlab.org/nlab/show/second-countable+space Accessed on 19th December 2023.

[10] Manjusha Majumdar and Arindam Bhattacharyya. *An Introduction to Smooth Manifolds.* Singapore, 1st edition, 2023.

[11] Michael Spivak. *Calculus on Manifolds.* Addison-Welsey Publishing Company, Brandeis University, 1965.

[12] George Gabriel Stokes. *1854 Smith's Prize Exam.* Cambridge, 1854.

[13] Loring W. Tu. *An Introduction to Manifolds.* 2nd edition, 2011.

[14] Élie Cartan. *Les Systèmes Différentiels Extérieurs et leurs Applications Géométriques.* Hermann, Paris, 1st edition, 1945.