Irrationality Measure

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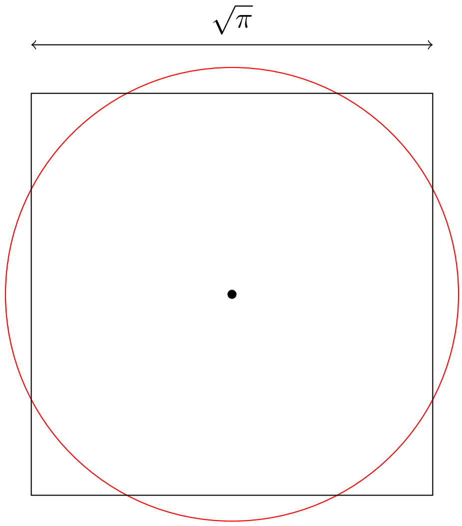
1 Introduction

Squaring the circle is a problem that fascinated ancient Greek mathematicians. The problem states: can one construct a square that has the area of a given circle with a finite number of steps using just a compass and straightedge. For example, the unit circle and square below both have an area of π.



In 1837, Pierre Wantzel proved that for this to be possible, π must be an algebraic number. Almost 50 years later, Ferdinand von Lindemann proved that π was transcendental, showing that this construction was impossible. In this essay we will discuss how the first transcendental numbers were constructed using sequences of rationals that achieved such good approximations that the numbers could not be algebraic.

We will extend this idea in the spirit of the well known Dirichlet's approximation theorem. For an irrational α, there exists infinitely many $\frac{p}{q}$ such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}.$$

We shall see that in fact, for most numbers, we cannot do much better than an exponent of 2.

2 Algebraic and transcendental numbers

The theory of transcendental numbers emerged as a profound extension of number theory in the 19th century. While algebraic numbers had been well studied, the existence of numbers beyond this was an open question until Joseph Liouville's work in 1844. Liouville's theorem provided a way to distinguish certain numbers as transcendental. By establishing a fundamental bound on the approximation of algebraic numbers by rationals, Liouville not only proved that transcendental numbers exist but also constructed the first explicit examples.

Definition 2.1. A number real number α is called **algebraic** if it satisfies an equation of the form

$$k_0 + k_1\alpha + k_2\alpha^2 + ... + k_n\alpha^n = 0$$

with integer coefficients, not all zero. A number that is not algebraic is called **transcendental**.

Definition 2.2. The **degree** of an algebraic number α is the smallest $n \in \mathbb{Z}^+$ such that α satisfies an equation of degree n.

We see that a rational number $\frac{p}{q}$ is algebraic with degree 1 since $qx - p = 0$.

Theorem 2.3 (Liouville approximation theorem)**.** *For any algebraic number α of degree $n > 1$ there exists $C(\alpha)$ such that*

$$\left|\alpha - \frac{p}{q}\right| > \frac{C}{q^n}$$

for all rationals $\frac{p}{q}$, $q > 0$.

Proof. (Baker 1975, p. 1). Let $f(x) \in \mathbb{Z}[x]$ be a polynomial of degree n for which $f(\alpha) = 0$. By the *Mean Value Theorem* see that for any rational we have

$$\left|f(\alpha) - f(p/q)\right| = \left|f'(d)\right| \left|\alpha - p/q\right|$$

for some d between p/q and α. Recalling that α is a root of the polynomial yields

$$\left|f(p/q)\right| = \left|f'(d)\right| \left|\alpha - p/q\right|.$$

Clearly the next step is to deal with the derivative, notice that we want to show $\left|\alpha - p/q\right| > 1/cq^n$. This is clearly true if $\left|\alpha - p/q\right| > 1$ thus we can assume that $\left|\alpha - p/q\right| < 1$. Since f is a polynomial we know that f' will be as well, therefore on some finite interval we guarantee that bounds exist. Let c be a positive integer such that $\left|f'(x)\right| < c$ whenever $|\alpha - x| < 1$. We also multiply by the integer q^n leaving us with

$$\left|q^n f(p/q)\right| < cq^n \left|\alpha - p/q\right|.$$

For any rational $f(p/q) \neq 0$ since otherwise α would satisfy an equation with degree less than n. Thus the integer $q^n f(p/q)$ is at least 1. To see that it is indeed an integer it suffices to substitute p/q into an arbitrary polynomial of degree n and multiply through by q^n. The result follows immediately by setting $C = \frac{1}{c}$.

□

With this result in hand, Liouville provided the first example of a transcendental number.

Proposition 2.4.

$$l = \sum_{m=1}^{\infty} \frac{1}{10^{m!}} \text{ is transcendental.}$$

Proof. (Liouville 1851) Suppose l is an algebraic number. Clearly l cannot be rational since it has a decimal expansion that is neither finite nor recurs. Thus l has degree $n > 1$. For $k \in \mathbb{Z}^+$ define p_k, q_k as:

$$p_k = 10^{k!}\Big(\frac{1}{10^{1!}} + \frac{1}{10^{2!}} + \cdots + \frac{1}{10^{k!}}\Big) \quad \text{and} \quad q_k = 10^{k!}.$$

Note that both p_k and q_k are integers. We can now estimate l by $\frac{p_k}{q_k}$ as follows

$$\begin{aligned}
\left|l - \frac{p_k}{q_k}\right| &= \sum_{m=k+1}^{\infty} \frac{1}{10^{m!}} \\
&= \frac{1}{10^{(k+1)!}}\Big(1 + \frac{1}{10^{k+2}} + \frac{1}{10^{k+2}10^{k+3}} + \frac{1}{10^{k+2}10^{k+3}10^{k+4}} + \cdots\Big) \\
&< \frac{1}{10^{(k+1)!}}\Big(1 + \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \cdots\Big) \\
&= \frac{10/9}{10^{(k+1)!}} \\
&< \frac{2}{10^{(k+1)!}}.
\end{aligned}$$

By the *Liouville approximation theorem*, there must be C such that for all k,

$$\frac{2}{10^{(k+1)!}} > \frac{C}{(10^{k!})^n}.$$

It then follows that

$$\frac{2}{C} > 10^{k!(k+1-n)}.$$

This is a contradiction for sufficiently large k since C is a fixed finite value, hence l is transcendental. □

In light of this example we define a class of numbers, the Liouville numbers, defined to be all those that can be approximated by rational numbers with extraordinary accuracy.

Definition 2.5. A number x is a **Liouville number** if for each $n \in \mathbb{Z}^+$ there exists integers p and q such that

$$0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^n} \quad and \quad q > 1.$$

Our goal is to generalise the method employed in proposition 2.4 to show that all Liouville numbers must be transcendental. Some care is required, it is not immediately obvious from the definition that a Liouville number cannot be rational.

Lemma 2.6. *All Liouville numbers are irrational.*

Proof. Suppose for a contradiction that x is a Liouville number and that $x = \frac{c}{d}$ where c, d are integers and $d > 0$.

$$\left| \frac{c}{d} - \frac{p}{q} \right| = \frac{|cq - dp|}{dq}.$$

It is clear we cannot have $|cq - dp| = 0$ as we would violate the left inequality in definition 2.5. Since we know it is an integer we have that $|cq - dp| \geq 1$.

$$\frac{|cq - dp|}{dq} \geq \frac{1}{dq}.$$

Recall that $q \geq 2$, thus $\frac{1}{q^n} \leq \frac{1}{2^{n-1}q}$. We then see that as long as $n > 1 + log_2(d)$ we will have

$$\frac{|cq - dp|}{dq} \geq \frac{1}{dq} \geq \frac{1}{2^{n-1}q} \geq \frac{1}{q^n}.$$

□

Lemma 2.7. *All Liouville numbers are transcendental.*

Proof. Let us suppose that x is a Liouville number that is algebraic of degree k. Since x is Liouville, for all $n \in \mathbb{Z}^+$ there exists $p, q \in \mathbb{Z}$ with $q > 1$ such that

$$0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^n}.$$

By lemma 2.6 we know that it must be irrational and so $k > 1$. This allows us to use the *Liouville approximation theorem* to guarantee that there exists some C such that

$$\left| x - \frac{p}{q} \right| > \frac{C}{q^k}$$

for all $p, q \in \mathbb{Z}$ where $q > 1$. It then follows from $q > 1$ that we must have

$$\frac{1}{C} > q^{n-k} \geq 2^{n-k}.$$

However, from the assumption that x has fixed degree k, we can choose $n \in \mathbb{Z}^+$ such that $2^n \geq \frac{2^k}{C}$. Leading to the contradiction that we must have

$$\frac{1}{C} > 2^{n-k} \geq \frac{1}{C}.$$

□

It seems we now have a way to check if some numbers are transcendental by checking definition 2.5. However, it is not immediately clear how we would check this for some arbitrary number, or if it is even useful to try. We have not proven the converse of the statement so it also remains to be seen if there exist transcendentals that are not Liouville. We have a general property for Liouville numbers, we utelise this to get a handle on the size of the set.

Theorem 2.8. *The set of Liouville numbers L has Lebesgue measure zero.*

Proof. (Oxtoby 1980, p. 8). Recall that definition 2.5 tells us that for all $n \in \mathbb{Z}^+$ we will have a pair p, q such that the inequality holds, hence we can write

$$L = \bigcap_{n=1}^{\infty} \bigcup_{q=2}^{\infty} \bigcup_{p=-\infty}^{\infty} \left(\left(\frac{p}{q} - \frac{1}{q^n}, \frac{p}{q} + \frac{1}{q^n} \right) \setminus \left\{ \frac{p}{q} \right\} \right).$$

Consider for $n > 2$ and $q \geq 2$ the sets

$$U_{n,q} = \bigcup_{p=-\infty}^{\infty} \left(\frac{p}{q} - \frac{1}{q^n}, \frac{p}{q} + \frac{1}{q^n} \right).$$

Since L is constructed as the intersection over all n, it is clear that we must have

$$L \subseteq \bigcup_{q=2}^{\infty} U_{n,q}.$$

For any two positive integers m and n we have

$$L \cap (-m, m) \subseteq \bigcup_{q=2}^{\infty} U_{n,q} \cap (-m, m) = \bigcup_{q=2}^{\infty} \Big[U_{n,q} \cap (-m, m) \Big] \subseteq \bigcup_{q=2}^{\infty} \bigcup_{p=-mq}^{mq} \left(\frac{p}{q} - \frac{1}{q^n}, \frac{p}{q} + \frac{1}{q^n} \right).$$

Thus $L \cap (-m, m)$ can be covered by a sequence of intervals. For $n > 2$ we can find an upper bound for the length.

$$\sum_{q=2}^{\infty} \sum_{p=-mq}^{mq} \frac{2}{q^n} = \sum_{q=2}^{\infty} \frac{2}{q^n} \sum_{p=-mq}^{mq} 1 = \sum_{q=2}^{\infty} \left(\frac{4m}{q^{n-1}} + \frac{2}{q^n} \right) \leq \sum_{q=2}^{\infty} \left(\frac{4m}{q^{n-1}} + \frac{q}{q^n} \right) = (4m+1) \sum_{q=2}^{\infty} \frac{1}{q^{n-1}}$$
$$\leq (4m+1) \int_1^{\infty} \frac{1}{x^{n-1}} \, dx = (4m+1) \left[\frac{x^{2-n}}{2-n} \right]_1^{\infty} = \frac{4m+1}{n-2}.$$

So for any choice of m we can find a sufficiently large n such that $L \cap (-m, m)$ is covered by a sequence of intervals with total length less than an arbitrary ε. □

Given that on $\mathbb{R}$ the transcendentals have full measure it is clear we are still far from any meaningful classification. Currently we have only considered numbers that have infinitely many good approximations for any fixed exponent of q. A potential improvement could be to specify a bound.

3 Irrationality Measure

Consider a real number α. The question of how the difference

$$\left|\alpha - \frac{p}{q}\right|$$

changes for choices of $p \in \mathbb{Z}$ and $q \in \mathbb{Z}^+$ is one of great importance in number theory and its applications. Of course, since $\mathbb{Q}$ is dense in $\mathbb{R}$ there will always be some p and q such that the difference is as small as we like. This is not particularly interesting and is why we will instead focus on how small we can make the difference relative to q.

We have already seen in the previous chapter a specific use of this, more generally we will consider

$$0 < \left|\alpha - \frac{p}{q}\right| < \frac{1}{q^\mu}$$

for different values of μ and determine if there exist finitely or infinitely many solutions $\frac{p}{q}$. We want to see how large we can make μ before α in a sense becomes badly approximated by rational numbers.

Definition 3.1. Let $\alpha \in \mathbb{R}$. The **irrationality measure** of α , denoted $\mu(\alpha)$, is the supremum over real $\mu > 0$ such that the above inequality is satisfied by an infinite number of integer pairs (p, q) with $q > 0$.

It follows that for all $\varepsilon > 0$ there exists some $C(\alpha) > 0$ such that for all rationals $\frac{p}{q}$ we have

$$\left|\alpha - \frac{p}{q}\right| > \frac{C}{q^{\mu+\varepsilon}}.$$

Since values larger than μ only admit finitely many rational solutions to the inequality, we guarantee that there will be some constant small enough to trump the best approximation. Note that this is almost identical to the statement of *Liouville approximation theorem.* Thus we have an upper bound for algebraic α of degree n, $\mu(\alpha) \leq n$. What about a lower bound?

Lemma 3.2. *Suppose $\alpha \in \mathbb{Q}$. Then $\mu(\alpha) = 1$.*

Proof. Suppose that $\alpha = \frac{a}{b}$ where a and b are coprime.

$$0 < \left|\alpha - \frac{p}{q}\right| = \frac{|aq - bp|}{bq}.$$

Observe that by Bézout's lemma the equation $aq - bp = 1$ has a solution (p_0, q_0) that can be computed using the extended Euclidean algorithm. We can now produce other solutions (p, q) by setting $p = p_0 + at$ and $q = q_0 + bt$ where $t \in \mathbb{Z}$. Thus we have infinitely many solutions (p, q) exist for

$$\left|\alpha - \frac{p}{q}\right| = \frac{1}{bq} \leq \frac{1}{q}.$$

It follows that $\mu(\alpha) \geq 1$.

Since $|aq - bp|$ is non zero we also get that for all $\varepsilon > 0$ and for any $0 < C \leq \frac{1}{b}$

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{1}{bq} > \frac{C}{q^{1+\varepsilon}}$$

for all rational $\frac{p}{q}$. Hence $\mu(\alpha) \leq 1$ and the lemma follows. □

Lemma 3.3. *Suppose $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then $\mu(\alpha) \geq 2$.*

Proof. This follows immediately from the *Dirichlet's approximation theorem* and was seen in analysis 1. See (Robinson 2022, p. 13). □

We see then that if α is an algebraic number of degree n we must have $2 \leq \mu(\alpha) \leq n$. It is natural to ask if we can improve these bounds. This question was first answered by Axel Thue, who in 1909 made the first improvement to Liouville's upper bound, $\mu(\alpha) \leq \frac{n}{2} + 1$. Over the course of the early 20^{th} century several more improvements were made to the upper bound. The matter was finally laid to rest in 1955 by Klaus Roth who proved the following.

Theorem 3.4 (Roth's theorem)**.** *For any $\varepsilon > 0$ and α algebraic, the inequality*

$$0 < \left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{2+\varepsilon}}$$

has only finitely many solutions.

These improvements presented a new challenge. We recall that in the proof of the Liouville approximation theorem our constant C was constructed by bounding the derivative of the polynomial, $\left|f'(x)\right| < \frac{1}{C}$. What is key here is that for a specific α the proof gives us an explicit bound that only depends α.

Lemma 3.5. *An explicit value for the constant in the Liouville approximation theorem is $C(\alpha) = (n^2(1+|\alpha|)^{n-1}H)^{-1}$. Where H is the maximum of the absolute values of the coefficients of f.*

Proof. Suppose $f(x) = a_0 + a_1x + \cdots + a_nx^n$. Set $H = \max\{|a_0|, |a_1|, \ldots, |a_n|\}$. Since we have that $|\alpha - x| < 1$ by the reverse triangle inequality we must have $|x| < 1+|\alpha|$, where importantly the right hand side is greater than 1. It then follows that

$$\left|f'(x)\right| < H + 2H(1+|\alpha|) + 3H(1+|\alpha|)^2 + \cdots + nH(1+|\alpha|)^{n-1} < n^2(1+|\alpha|)^{n-1}H.$$

□

This shows that Liouville's result was effective. In contrast, the later improvements were not. The issue is already present in Thue's improvement, in order to show that there are not too many good approximations $\frac{p}{q}$ Thue starts with an incredibly good one $\frac{p_0}{q_0}$. He shows that this very close approximation excludes other similar or better ones. However, it can also be the case that this $\frac{p_0}{q_0}$ simply does not exist, which would also give the needed result. The ineffectivity stems from the fact that we do not know from which case the conclusion is from. A good introduction to Thue's proof can be found in chapter 5 of Silverman & Tate (1992) which proves the result for $\alpha = \sqrt[3]{b}$. In 1964, Alan Baker found an effective bound for the case $b = 2$.

Theorem 3.6. *Baker (1964) For all rationals p/q ($q > 0$) we have*

$$\left|\sqrt[3]{2} - \frac{p}{q}\right| > \frac{10^{-6}}{q^{2.955}}.$$

Let us take stock of where we are for some $\alpha \in \mathbb{R}$.

$$\begin{cases} \mu(\alpha) = 1 & \alpha \text{ is rational} \\ \mu(\alpha) = 2 & \alpha \text{ is algebraic of degree n} > 1 \\ \mu(\alpha) \geq 2 & \alpha \text{ is transcendental} \end{cases}$$

We now have a much better classification, all that remains to be seen is exactly how many transcendentals are mixed in with the algebraic irrationals.

Lemma 3.7. *The set of real numbers with irrationality measure greater than two has measure zero.*

Proof. It suffices to consider $x \in (0,1) \cap \mathbb{R} \setminus \mathbb{Q}$ since we can just shift rational approximations by some integer. The inequality is satisfied by infinitely many $\frac{p}{q}$ for some $\varepsilon > 0$.

$$\frac{p}{q} - \frac{1}{q^{2+\varepsilon}} < x < \frac{p}{q} + \frac{1}{q^{2+\varepsilon}}.$$

We can reformulate this inequality by considering

$$E_q = (0,1) \cap \bigcup_{p=0}^{q} \Big(\frac{p}{q} - \frac{1}{q^{2+\varepsilon}}, \frac{p}{q} + \frac{1}{q^{2+\varepsilon}} \Big).$$

Note that the inequality being satisfied for infinitely many $\frac{p}{q}$ is equivalent to $x \in E_q$ for infinitely many q. It is clear that

$$\lambda(E_q) = \frac{2(q+1)}{q^{2+\varepsilon}} \text{ and so for all } \varepsilon > 0 \text{ the sum } \sum_{q=1}^{\infty} \lambda(E_q) < \infty.$$

It now follows by the first Borel-Cantelli lemma that the set

$$E = \{x \in (0,1) : x \in E_q \text{ for infinitely many q } \}$$

is a set of measure zero. □

We have shown that almost all transcendental numbers have measure 2, the aim of the next chapter is to provide a concrete example.

4 Continued Fractions and e

The number e is one of the most important mathematical constants, appearing naturally in many different fields of mathematics. In this chapter, we explore the continued fraction representation of e and its role in estimating its irrationality measure. Which we will show to be $\mu(e) = 2$. Whilst irrationality was already known to Euler in the 18th century, it was only in 1873 that Hermite was able to prove that it was also transcendental.

Theorem 4.1. *e is a transcendental number*

Proof. The proof given here follows the main ideas in (Havil 2012, p. 191) which seeks to replicate Hermite's proof. Alterations have been made to the derivative cases in the latter half of the proof, these were motivated by (Baker 1975, p. 4). Suppose $f(x)$ is any polynomial in x and define

$$F(x) = \sum_{k=0}^{\infty} f^{(k)}(x)$$

This is clearly a polynomial of the same degree as $f(x)$. We see also that $f(x) = F(x) - F'(x)$ and hence

$$\begin{aligned} \frac{d}{dx}(e^{-x}F(x)) &= e^{-x}F'(x) - e^{-x}F(x) \\ &= -e^{-x}(F(x) - F'(x)) = -e^{-x}f(x). \end{aligned}$$

This yields

$$e^x \int_0^x e^{-t} f(t)\, dt = e^x[-e^{-t}F(t)]_0^x = e^x F(0) - F(x). \tag{1}$$

For a contradiction we will assume that e is algebraic. So we have some polynomial of degree n with integer coefficients ($a_0 \neq 0$) such that

$$a_0 + a_1 e + a_2 e^2 + \cdots + a_n e^n = 0.$$

Evaluating (1) at k, multiplying by a_k and summing over k gives us

$$\sum_{k=0}^{n} a_k e^k \int_0^k e^{-t} f(t)\, dt = F(0) \sum_{k=0}^{n} a_k e^k - \sum_{k=0}^{n} a_k F(k).$$

We now use the assumption to reach

$$\sum_{k=0}^{n} a_k F(k) = -\sum_{k=0}^{n} a_k e^k \int_0^k e^{-t} f(t)\, dt.$$

Our aim now is to choose a polynomial that will eventually lead us to a contradiction, for a prime $p > n + |a_0|$ define $f(t)$ of degree $np + p - 1$ by

$$f(t) = \frac{t^{p-1}(t-1)^p(t-2)^p \dots (t-n)^p}{(p-1)!}.$$

Note that on $[0, n]$ we have the following crude bound on the size of $f(t)$.

$$\big|f(t)\big| \leq \frac{n^{p-1}(n^p n^p \dots n^p)}{(p-1)!} = \frac{n^{np+p-1}}{(p-1)!}.$$

Thus

$$\begin{aligned}
\left|\sum_{k=0}^{n} a_k e^k \int_0^k e^{-t} f(t)\, dt\right| &\leq \sum_{k=0}^{n} |a_k|\, e^k \int_0^k e^{-t} \big|f(t)\big|\, dt \\
&\leq e^k \sum_{k=0}^{n} |a_k| \int_0^k e^{-t} \frac{n^{np+p-1}}{(p-1)!}\, dt \\
&= e^k \frac{n^{np+p-1}}{(p-1)!} \sum_{k=0}^{n} |a_k| \int_0^k e^{-t}\, dt = \frac{n^{np+p-1}}{(p-1)!} \sum_{k=0}^{n} |a_k|\, (e^k - 1).
\end{aligned}$$

Since the coefficients a_k and degree n are fixed we see that we can choose a large enough p to ensure the factorial in the denominator reduces the expression below 1. Our goal is now clear, if we can show that $\sum_{k=0}^{n} a_k F(k)$ has absolute value ≥ 1 we will have the contradiction we desire. We need to evaluate $F(k)$ for $0 \leq k \leq n$. Let us first consider $F(0)$.

We decompose $f(t) = a(t)b(t)$ where

$$a(t) = \frac{t^{p-1}}{(p-1)!} \quad \text{and} \quad b(t) = (t-1)^p \dots (t-n)^p.$$

It is clear that $a^{(i)}(0) = 0$ for all i except $i = p - 1$ where we have $a^{(p-1)}(0) = 1$. Applying the general Leibniz rule for differentiation to this product yields

$$f^{(j)}(t) = \sum_{i=0}^{j} \binom{j}{i} a^{(i)}(t) b^{(j-i)}(t).$$

Hence

$$F(0) = \sum_{j=0}^{\infty} f^{(j)}(0) = \sum_{j=0}^{\infty}\sum_{i=0}^{j} \binom{j}{i} a^{(i)}(0) b^{(j-i)}(0) = \sum_{j=p-1}^{\infty} \binom{j}{p-1} b^{(j-p+1)}(0).$$

We see that for all $i \geq 1$, $b^{(i)}(t)$ is a multiple of p. However, in the case when $j = p-1$ we have $b(0) = (-1)^p \dots (-n)^p$. Since we chose $p > n$ we must have that $b(0)$ is not divisible by p. Thus $F(0)$ is an integer that does not have p as a factor. We can proceed similarly for the other values of k. We decompose in a similar fashion, $f(t) = a(t)b(t)$ where we have removed $(t-k)^p$ from $b(t)$.

$$a(t) = \frac{(t-k)^p}{(p-1)!} \quad \text{and} \quad b(t) = t^{p-1}(t-1)^p \dots (t-n)^p.$$

Note that here $a^{(i)}(k) = 0$ for all i except $i = p$ where $a^{(p)}(k) = p$. We now see that for $1 \leq k \leq n$ we have

$$F(k) = \sum_{j=0}^{\infty} f^{(j)}(k) = \sum_{j=0}^{\infty}\sum_{i=0}^{j} \binom{j}{i} a^{(i)}(k) b^{(j-i)}(k) = p \sum_{j=p}^{\infty} \binom{j}{p} b^{(j-p)}(k).$$

Showing that $F(k)$ is an integer that has p as one of its factors. We can now consider $\sum_{k=0}^{n} a_k F(k) = a_0 F(0) + \sum_{k=1}^{n} a_k F(k)$. Since $p > |a_0|$ we have that p cannot divide $a_0 F(0)$. So if we were to divide the entire expression by p we would be summing a fraction and an integer, showing that $\sum_{k=0}^{n} a_k F(k)$ is not 0. Since it is also an integer, it must have absolute value ≥ 1. $\square$

A key tool in the study of diophantine approximation is the notion of continued fractions, which provide a way of generating the best possible rational approximations to a given number. The connection between continued fractions and the irrationality measure is visible through the bound

$$\left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}} < \frac{1}{a_{n+1} q_n^2}.$$

This inequality suggests that we may be able to find estimates from the irrationality measure by analysing the growth rate of q_n.

Definition 4.2. A simple continued fraction is of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\dots + \frac{1}{a_n}}}}$$

where the a_i is a (possibly infinite) sequence of positive integers and are called the partial quotients. In a more compact form we can express the simple continued fraction as $[a_0; a_1, \dots, a_n]$.

Notice that each truncation will yield a new rational number, by considering the first few we begin to see a pattern.

$$T_0 = \frac{a_0}{1},$$

$$T_1 = a_0 + \frac{1}{a_1} = \frac{a_0 a_1 + 1}{a_1},$$

$$T_2 = a_0 + \frac{1}{a_1 + \frac{1}{a_2}} = \frac{a_0(a_1 a_2 + 1) + a_2}{a_1 a_2 + 1}.$$

We see that both the numerator and denominator seem to depend on the previous fraction, indicating we should try and define a recurrence relation for them. From the above one can guess that

$$p_{-1} = 1, \quad p_0 = a_0 \quad \text{and} \quad q_{-1} = 0, \quad q_0 = 1,$$
$$\begin{cases} p_{k+1} = a_{k+1}p_k + p_{k-1}, \\ q_{k+1} = a_{k+1}q_k + q_{k-1}, \end{cases} \quad \text{for } 0 \leq k \leq n-1.$$

Lemma 4.3. *The fraction* $\frac{p_k}{q_k}$ *is the truncation* T_k *for all* k.

Proof. (Rockett et al. 1992, p. 2) Suppose that this is true for all truncations of order $\leq k$. Consider T_{k+1}, we can reduce this to our assumption by considering a new last term of $a_k + \frac{1}{a_{k+1}}$. This gives us

$$\frac{(a_k + \frac{1}{a_{k+1}})p_{k-1} + p_{k-2}}{(a_k + \frac{1}{a_{k+1}})q_{k-1} + q_{k-2}}$$

where the quantities $p_{k-1}, p_{k-2}, q_{k-1}$ and q_{k-2} are known by our assumption. This fraction is the same as T_{k+1} so we see that

$$T_{k+1} = \frac{(a_k + \frac{1}{a_{k+1}})p_{k-1} + p_{k-2}}{(a_k + \frac{1}{a_{k+1}})q_{k-1} + q_{k-2}} = \frac{a_k p_{k-1} + p_{k-2} + \frac{p_{k-1}}{a_{k+1}}}{a_k q_{k-1} + qk - 2 + \frac{q_{k-1}}{a_{k+1}}} = \frac{a_{k+1}p_k + p_{k-1}}{a_{k+1}q_k + q_{k-1}} = \frac{p_{k+1}}{q_{k+1}}.$$

□

We call the fractions $\frac{p_k}{q_k}$ the convergents. A simple algebraic check gives us that they also satisfy

$$p_k q_{k-1} - p_{k-1}q_k = (-1)^{k-1} \text{ or equivalently } \frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}} = \frac{(-1)^{k-1}}{q_k q_{k-1}}.$$

One can use these to show that the convergents form a Cauchy sequence. The oscillating sign shows they approximate the limit alternately from above and below.

Lemma 4.4. *For* $k \geq 1$ *we have that* $q_k p_{k-2} - p_k q_{k-2} = (-1)^{k-1} a_k$.

Proof. From the recursive relations we found above we see that

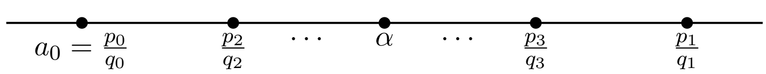
$$\begin{aligned} q_k p_{k-2} - p_k q_{k-2} &= p_{k-2}(a_k q_{k-1} + q_{k-2}) - q_{k-2}(a_k p_{k-1} + p_{k-2}) \\ &= a_k(q_{k-1}p_{k-2} - p_{k-1}q_{k-2}) \\ &= a_k(-1)^{k-1}. \end{aligned}$$

Where the final equality follows from the previous lemma. □

Definition 4.5. We call a rational $\frac{a}{b}$ a **best approximation** of α if $|b\alpha - a| < |q\alpha - p|$ for any rational $\frac{p}{q} \neq \frac{a}{b}$ where $0 < q \leq b$.

Theorem 4.6. *Every best approximation to* α *is a convergent of the simple continued fraction of* α.

Proof. We follow the idea set out in (Lang 2012, p. 9). Let $\alpha = [a_0; a_1, a_2, \ldots]$. From before we know that the convergents approximate α from above and below



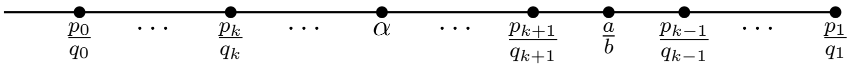
Suppose $\frac{a}{b}$ is not a convergent. We will deal with different cases as to where it could be. To start with, suppose $\frac{a}{b} < \frac{p_0}{q_0}$.

$$|\alpha - a_0| < \left|\alpha - \frac{a}{b}\right| \leq b\left|\alpha - \frac{a}{b}\right| = |b\alpha - a|\,,$$

which contradicts that $\frac{a}{b}$ is a best approximation. Now suppose it lies at the other end, that $\frac{a}{b} > \frac{p_1}{q_1}$.

$$|b\alpha - a| = b\left|\alpha - \frac{a}{b}\right| > b\left|\frac{p_1}{q_1} - \frac{a}{b}\right| \geq \frac{b}{q_1 b} = \frac{1}{q_1} = \frac{1}{a_1}.$$

By the continued fraction expansion we have that $|\alpha - a_0| < \frac{1}{a_1}$. So in tandem with the above we again get a contradiction $|\alpha - a_0| < |b\alpha - a|$. We are left with a single case, that $\frac{a}{b}$ lies between two convergents. With out loss of generality assume we are in the situation below.



Then
$$\frac{1}{q_k q_{k-1}} = \left|\frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}}\right| > \left|\frac{a}{b} - \frac{p_{k-1}}{q_{k-1}}\right| \geq \frac{1}{bq_{k-1}}.$$

We have then that $q_k < b$. With a similar argument we see that

$$\frac{1}{q_k q_{k+1}} > \left|\alpha - \frac{p_k}{q_k}\right|$$

Leading to $|q_k\alpha - p_k| < \frac{1}{q_{k+1}}$. It remains to find an estimate on how well α is approximated by $\frac{a}{b}$ like before.

$$|b\alpha - a| = b\left|\alpha - \frac{a}{b}\right| > b\left|\frac{p_{k+1}}{q_{k+1}} - \frac{a}{b}\right| \geq \frac{1}{q_{k+1}}.$$

Combining these together gives

$$|q_k\alpha - p_k| < \frac{1}{q_{k+1}} < |b\alpha - a|$$

which is a contradiction since $q_k < b$.

□

We note that the converse of this is also true, and can be proven by induction. We will not need this result but refer the interested reader to (Lang 2012, p. 9). This theorem gives us an essential tool in dealing with convergents, one we will put to immediate use. Recall that our aim is to study $\left|\alpha - \frac{p}{q}\right|$, we can show that any $\frac{p}{q}$ that admits a sufficiently good approximation is necessarily a convergent.

Lemma 4.7. *For integers p, q where $q > 0$, the inequality*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2}$$

implies that $\frac{p}{q}$ is a convergent of the continued fraction of α.

Proof. It suffices to show that $\frac{p}{q}$ is a best approximation of α . Suppose we have $\frac{c}{d}$ where $d > 0$ such that

$$|d\alpha - c| \leq |q\alpha - p| < \frac{1}{2q}.$$

Comparing our two estimates gives that

$$\frac{1}{qd} \leq \left|\frac{c}{d} - \frac{p}{q}\right| \leq \left|\alpha - \frac{c}{d}\right| + \left|\alpha - \frac{p}{q}\right| < \frac{1}{2qd} + \frac{1}{2q^2} = \frac{q+d}{2q^2 d}.$$

From which we can conclude that $q < d$ and so $\frac{p}{q}$ is a best approximation to α. □

We are now in a position to provide a concrete link between the simple continued fraction of α and its irrationality measure.

Theorem 4.8. *The irrationality measure of an irrational number α with simple continued fraction expansion $\alpha = [a_0; a_1, a_2, \dots]$ and convergents $\frac{p_n}{q_n}$ is given by*

$$\mu(\alpha) = 1 + \limsup_{n\to\infty} \frac{\log q_{n+1}}{\log q_n} = 2 + \limsup_{n\to\infty} \frac{\log a_{n+1}}{\log q_n}.$$

Proof. A proof of this result is presented in (Sondow 2004, p. 5), though with some details omitted. We seek to reconstruct this proof in totality. Define λ_n by the equation

$$\left|\alpha - \frac{p_n}{q_n}\right| = \frac{1}{q_n^{\lambda_n}}.$$

Recall that the convergents approximate α alternately from above and below. As they satisfy the identity $\frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}} = \frac{(-1)^{k-1}}{q_k q_{k-1}}$ we see that we must have $\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}}$. It follows that

$$\frac{1}{q_n^{\lambda_n}} = \left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}} < \frac{1}{q_n^2}$$

which shows that $\lambda_n > 2$. We have that $\lambda := \limsup_{n\to\infty} \lambda_n \geq 2$.

Suppose that $\mu(\alpha) < \infty$, we claim that $\lambda \leq \mu(\alpha)$. Indeed, given $\varepsilon > 0$ there exists $Q \in \mathbb{N}$ such that for all $p \in \mathbb{Z}, q \geq Q$ we have

$$\left|\alpha - \frac{p}{q}\right| > \frac{1}{q^{\mu(\alpha)+\varepsilon}}.$$

We now choose $N \in \mathbb{N}$ such that $q_N > Q$. It follows that for all $n > N$,

$$\frac{1}{q_n^{\lambda_n}} = \left|\alpha - \frac{p_n}{q_n}\right| > \frac{1}{q_n^{\mu(\alpha)+\varepsilon}}.$$

Implying that $\lambda_n < \mu(\alpha) + \varepsilon$ for all $n > N$. From this we see that

$$\lambda = \limsup_{n\to\infty} \lambda_n \leq \mu(\alpha) + \varepsilon$$

for all $\varepsilon > 0$.

Now suppose that $\lambda < \infty$, we claim that $\mu(\alpha) \leq \lambda$. Note that for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\lambda_n < \lambda + \varepsilon$ for all $n > N$. We argue for a contradiction, suppose $\lambda + \varepsilon < \mu(\alpha)$. Then there are infinitely many solutions $\frac{p}{q}$ to

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{q^{\lambda+\varepsilon}}.$$

Restricting to solutions where $q > \max(2^{1/\varepsilon}, q_N)$ yields

$$\left|\alpha - \frac{p}{q}\right| \le \frac{1}{q^\lambda q^\varepsilon} < \frac{1}{2q^\lambda} \le \frac{1}{2q^2}.$$

It follows from lemma 4.7 that these $\frac{p}{q}$ must be a convergent of α. We have $\frac{p}{q} = \frac{p_n}{q_n}$ for some $n > N$. But this implies that

$$\frac{1}{q_n^{\lambda_n}} = \left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n^{\lambda+\varepsilon}}.$$

Which contradicts that $\lambda_n < \lambda + \varepsilon$ for $n > N$. Hence $\mu(\alpha) \le \lambda + \varepsilon$ for all $\varepsilon > 0$. We conclude that

$$\limsup_{n\to\infty} \lambda_n = \mu(\alpha).$$

Note that since $\frac{p_n}{q_n}$ are convergents so by lemma 4.4 we have

$$\left|\alpha - \frac{p_n}{q_n}\right| > \left|\frac{p_{n+2}}{q_{n+2}} - \frac{p_n}{q_n}\right| = \frac{a_{n+2}}{q_{n+2}q_n} = \frac{a_{n+2}}{(a_{n+2}q_{n+1} + q_n)q_n} \ge \frac{1}{(q_{n+1} + q_n)q_n} > \frac{1}{2q_nq_{n+1}}.$$

Where the second inequality follows by dividing through by a_{n+2} and using the fact it is greater than or equal to 1. We have that

$$\frac{1}{2q_nq_{n+1}} < \frac{1}{q_n^{\lambda_n}} < \frac{1}{q_nq_{n+1}}.$$

Taking logs and dividing through by $\log q_n$ yields

$$1 + \frac{\log q_{n+1}}{\log q_n} < \lambda_n < 1 + \frac{\log 2}{\log q_n} + \frac{\log q_{n+1}}{\log q_n},$$

thus

$$\mu(\alpha) = 1 + \limsup_{n\to\infty} \frac{\log q_{n+1}}{\log q_n}.$$

The second formula follows immediately since $q_{n+1} = a_{n+1}q_n + q_{n-1} = a_{n+1}q_n(1 + o(1))$. □

Lemma 3.3 gave us a lower bound for $\mu(\alpha)$, 2. Thus if we can show that the limsup tends to zero then we would immediately know that the irrationality measure of α is 2.

Lemma 4.9. *If the partial quotients of α satisfy $a_n = e^{o(n)}$ as $n \to \infty$ then $\mu(\alpha) = 2$.*

Proof. Since $a_i \ge 1$ for all $i \in \mathbb{N}$ and $q_{k+1} = a_{k+1}q_k + q_{k-1}$ we see that $q_n \ge F_n$ where F_n is the n^{th} Fibonacci number. Recall from Binet's formula that F_n is asymptotic to φ^n, where φ is the golden ratio. Hence $\log F_n$ is asymptotic to $n \log \varphi$. It follows that

$$0 \le \mu(\alpha) - 2 \le \limsup_{n\to\infty} \frac{\log a_{n+1}}{\log q_n} \le \limsup_{n\to\infty} \frac{\log a_{n+1}}{\log F_n} = 0.$$

□

We are now at a point where we have that e is transcendental and also a way to calculate the irrationality measure from the continued fraction. We would like to determine what the coefficients are:

$$e = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}}$$

Clearly $a_0 = 2$, rearranging gives us that

$$\frac{1}{e-2} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}$$

We have that $\frac{1}{e-2} = 1.3922\dots$ and so $a_1 = 1$. By repeating this computation repeatedly we begin to see a pattern emerge and we can conjecture that

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \dots].$$

This result was originally proven by Euler, the most common proof involves the continued fraction for $\tanh(x)$. The proof presented here is from Cohn (2006) which seeks to replicate the argument by Hermite using Padé approximants. The reason for this choice is largely because it relates nicely to our proof of e being transcendental as we utelise similar polynomials. However, we also have the benefit that the proof only relies on properties of e and does not need much else.

A Padé approximant to e^z of type (m, n) is a function $\frac{p(z)}{q(z)}$ where p and q are polynomials with degree m and n respectively. We have that the first $m + n + 1$ coefficients in the Taylor series of $\frac{p(z)}{q(z)}$ agree with those for e^z. That is, as $z \to 0$, the rational function satisfies

$$\frac{p(z)}{q(z)} = e^z + O(z^{m+n+1}).$$

Equivalently we could ask for the function

$$\frac{q(z)e^z - p(z)}{z^{m+n+1}}$$

to be holomorphic.

The Padé approximant of type (m, n) is unique, consider if $\frac{r(z)}{s(z)}$ is another. We see that

$$\frac{p(z)}{q(z)} - \frac{r(z)}{s(z)} = \frac{p(z)s(z) - q(z)r(z)}{q(z)s(z)} = O(z^{m+n+1}).$$

Note that $p(z)s(z) - q(z)r(z)$ has degree at most $m + n$ but vanishes to order $m + n + 1$, this gives us that it must be 0 and thus the two approximants are the same.

The Padé approximants give us a new way to approximate a power series. Since we wish to approximate e, we should set $z = 1$ in the Padé approximants for e^z. Take the approximant of type $(1, 2)$;

$$r_{1,2}(z) = \frac{1 + \frac{1}{3}x}{1 - \frac{2}{3}x + \frac{1}{6}x^2}$$

We now see that

$$r_{1,2}(1) = \frac{8}{3} = 2 + \frac{2}{3} = 2 + \cfrac{1}{1 + \cfrac{1}{2}} = [2; 1, 2].$$

In fact, following the same process for other types reveals

$$\begin{aligned}
r_{1,1}(1) &= [2;1],\\
r_{2,1}(1) &= [2;1,2,1],\\
r_{2,2}(1) &= [2,1,2,1,1],\\
r_{2,3}(1) &= [2,1,2,1,1,4],\\
r_{3,3}(1) &= [2,1,2,1,1,4,1,1].
\end{aligned}$$

It appears that the type (n,n),$(n+1,n)$ and $(n,n+1)$ Padé approximants are giving truncations of the expected simple continued fraction. Recall that for a function to be holomorphic we must be able to express it as a suitable integral. Hermite realised that the numerical pattern shown above could be linked to the simple continued fractions by means of some specific integrals.

Lemma 4.10. *Let $r(x)$ be a polynomial of degree k. Then there are polynomials $q(z)$ and $p(z)$ of degree at most k such that*

$$\int_0^1 r(x)e^{zx}dx = \frac{q(z)e^z - p(z)}{z^{k+1}}.$$

With

$$\begin{aligned}
q(z) &= r(1)z^k - r'(1)z^{k-1} + r''(1)z^{k-2} - \dots\\
p(z) &= r(0)z^k - r'(0)z^{k-1} + r''(0)z^{k-2} - \dots
\end{aligned}$$

Proof. This follows directly from repeated integration by parts. □

Given that we want polynomials $p(z)$ and $q(z)$ of degree m and n respectively, we take $k = m+n$ in the lemma above. Using the two explicit formulas above for p and q and the fact that we require $\deg(p) \leq m$, $\deg(q) \leq n$. We deduce that $r(x)$ must have a root of order n at $x = 0$ and a root of order m at $x = 1$. Up to a scaling constant we must have

$$r(x) = Ax^n(x-1)^m.$$

We make a choice[1] of $A = \frac{1}{n!}$. Setting $z = 1$ now yields

$$\int_0^1 \frac{x^n(x-1)^m}{n!}e^x dx = q(z)e - p(z).$$

Let us now consider again the continued fraction we conjectured previously, albeit in a slightly different form.

$$[1;0,1,1,2,1,1,4,1,1,6,1,1,8,\dots].$$

This is not in the form of a simple continued fraction however it can be easily seen that it is equivalent to the one we had before. The reason for this alteration is that it makes it easier to define the pattern. We have $a_{3i+1} = 2i$ and $a_{3i} = a_{3i+2} = 1$. The recurrence relations are thus given by

$$\begin{aligned}
p_{3k} &= p_{3k-1} + p_{3k-2}, & q_{3k} &= q_{3k-1} + q_{3k-2},\\
p_{3k+1} &= 2kp_{3k} + p_{3k-1}, & q_{3k+1} &= 2kq_{3k} + q_{3k-1},\\
p_{3k+2} &= p_{3k+1} + p_{3k}, & q_{3k+2} &= q_{3k+1} + q_{3k}.
\end{aligned}$$

[1] Any choice works since we can simply rescale our functions however this particular choice was made because it gives a cleaner solution. Note that $\frac{d^n}{dx^n}\left(\frac{x^n(x-1)^n}{n!}\right)$ has integral coefficients.

We calculate the first few convergents to be

k	0	1	2	3	4	5	6	7	8
p_k	1	1	2	3	8	11	19	87	106
q_k	1	0	1	1	3	4	7	32	39

Note that $\frac{p_1}{q_1}$ is undefined but this is not an issue since we are just interested in showing $\lim_{k\to\infty} \frac{p_k}{q_k} = e$.

We define the following three integrals, motivated by our discussion of the Padé approximants.

$$A_n = \int_0^1 \frac{x^n(x-1)^n}{n!} e^x \, dx,$$

$$B_n = \int_0^1 \frac{x^{n+1}(x-1)^n}{n!} e^x \, dx,$$

$$C_n = \int_0^1 \frac{x^n(x-1)^{n+1}}{n!} e^x \, dx.$$

It is simple to see that $A_0 = e-1$, $B_0 = 1$ and $C_0 = 2-e$. It can also be seen that $q_0 e - p_0 = e-1$, $p_1 - q_1 e = 1$ and $p_2 - q_2 e = 2 - e$. This suggests:

Lemma 4.11. *For $n \geq 0$, $A_n = q_{3n}e - p_{3n}$, $B_n = p_{3n+1} - q_{3n+1}e$, $C_n = p_{3n+2} - q_{3n+2}e$.*

Proof. We can reformulate these expressions just in terms of integrals by using the 6 recurrence relations from before. Consider:

$$-B_{n-1} = q_{3n-2}e - p_{3n-2},$$
$$-C_{n-1} = q_{3n-1}e - p_{3n-1}.$$

We see that

$$A_n = q_{3n}e - p_{3n} = (q_{3n-2} + q_{3n-1})e - (p_{3n-2} + p_{3n-1}) = -B_{n-1} - C_{n-1}.$$

Similar calculations can be done to reach

$$A_n = -B_{n-1} - C_{n-1},$$

$$B_n = -2nA_n + C_{n-1},$$

$$C_n = B_n - A_n.$$

It now suffices to check these three relations. The third one is clear. To prove the first we apply integration by parts to see

$$A_n = \int_0^1 \frac{x^n(x-1)^n}{n!} e^x \, dx = -\int_0^1 \frac{x^n(x-1)^{n-1}}{(n-1)!} e^x \, dx - \int_0^1 \frac{x^{n-1}(x-1)^n}{(n-1)!} e^x \, dx = -B_{n-1} - C_{n+1}.$$

For the second consider

$$\frac{d}{dx}\left(\frac{x^n(x-1)^{n+1}}{n!} e^x\right) = \frac{nx^{n-1}(x-1)^{n+1}}{n!} e^x + \frac{(n+1)x^n(x-1)^n}{n!} e^x + \frac{x^n(x-1)^{n+1}}{n!} e^x$$
$$= 2n\frac{x^n(x-1)^n}{n!} e^x - \frac{x^{n-1}(x-1)^n}{(n-1)!} e^x + \frac{x^{n+1}(x-1)^n}{n!} e^x.$$

Where the second equality comes from using $(x-1)^{n+1} = (x-1)^n(x-1)$ and expanding. Integrating both sides yields $2nA_n - C_{n-1} + B_n = 0$. □

Theorem 4.12. $e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, \ldots, 1, 2n, 1, \ldots]$.

Proof. Cohn (2006) The integrals A_n, B_n and C_n tend to 0 as $n \to \infty$ since we are integrating over $[0, 1]$. It follows from lemma 4.11 that

$$\lim_{k \to \infty} q_k e - p_k = 0.$$

Thus for $k \geq 2$ we have

$$\lim_{k \to \infty} \frac{p_k}{q_k} = e.$$

□

It follows immediately from Lemma 4.9 that the irrationality measure of e must be 2 due to the partial quotients only growing linearly.

5 A bound for π

In the previous chapter we saw that we could analyse the irrationality measure of e precisely by considering the continued fraction. Can we apply this approach to other known transcendental constants?

It can be shown that π is transcendental[2] in a similar fashion to our proof for e. However attempting to compute the first few partial quotients of the continued fractions yields

$$\pi = [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, \ldots].$$

Where there is no obvious pattern, currently it is believed that none exists. This prevents us from using our previous approach, this is the case for most constants.

Unlike e (and related constants) most natural[3] transcendentals have poorly behaved simple continued fraction from which is difficult to gain much information from. For this reason we require a different approach, namely, Beuker's treatment of Apéry's proof of the irrationality of $\zeta(3)$. The method we give here builds on the outline given in (Borwein & Borwein 1987, p. 366). We will provide an upper bound for the irrationality measure of $\zeta(2)$ and consequently one for π.

Lemma 5.1. *Let r and s be nonnegative integers.*

$$\int_0^1 \int_0^1 \frac{x^r y^s}{1 - xy} \, dx \, dy = \begin{cases} \frac{n}{d_r^2} \quad \text{for some } n \in \mathbb{Z}, & r > s, \\ \zeta(2) - \frac{1}{1^2} - \frac{1}{2^2} - \cdots - \frac{1}{r^2} & r = s > 0, \\ \zeta(2) & r = s = 0. \end{cases}$$

Where $d_r = \mathrm{LCM}(1, 2, \ldots, r)$.

Proof. We see that

$$\int_0^1 \int_0^1 \frac{x^r y^s}{1 - xy} \, dx \, dy = \sum_{n=0}^{\infty} \int_0^1 \int_0^1 x^{n+r} y^{n+s} \, dx \, dy = \sum_{n=0}^{\infty} \left(\frac{1}{n + r + 1} \frac{1}{n + s + 1} \right).$$

[2] This proof has been omitted, we refer the interested reader to (Baker 1975, p. 5).

[3] Natural in the sense that is has not been artificially created for this specific purpose.

If $r = s = 0$ it is clear this evaluates to $\zeta(2)$. If $r = s > 0$ we see that

$$\int_0^1 \int_0^1 \frac{x^r y^s}{1-xy}\, dx\, dy = \sum_{n=1}^{\infty} \left(\frac{1}{n+r}\right)^2 = \zeta(2) - \frac{1}{1^2} - \frac{1}{2^2} - \cdots - \frac{1}{r^2}.$$

Now suppose $r > s$.

$$\begin{aligned}\int_0^1 \int_0^1 \frac{x^r y^s}{1-xy}\, dx\, dy &= \sum_{n=0}^{\infty} \left(\frac{1}{n+r+1}\frac{1}{n+s+1}\right) \\ &= \sum_{n=0}^{\infty} \frac{1}{r-s}\left(\frac{1}{(n+s+1)} - \frac{1}{(n+r+1)}\right) \\ &= \frac{1}{r-s}\left(\frac{1}{s+1} + \frac{1}{s+2} + \cdots + \frac{1}{r}\right) = \frac{n}{d_r^2}.\end{aligned}$$

□

Theorem 5.2. $\zeta(2) = \frac{\pi^2}{6}$ *is irrational.*

Proof. (Borwein & Borwein 1987, p. 366) We begin by considering the Legendre-polynomial on $[0, 1]$, defined as

$$p_n(x) = \frac{1}{n!}\left\{\frac{d}{dx}\right\}^n x^n(1-x)^n$$

In order to be able to apply the previous lemma we reformulate this as a polynomial of degree n, note that we will have integer coefficients.

$$p_n(x) = \frac{1}{n!}\left\{\frac{d}{dx}\right\}^n \sum_{k=0}^{n} \binom{n}{k}(-1)^k x^{n+k} = \sum_{k=0}^{n} \binom{n}{k}\frac{(n+k)!}{n!k!}(-1)^k x^k.$$

We now consider the following integral

$$I_n = \int_0^1 \int_0^1 \frac{(1-y)^n p_n(x)}{1-xy}\, dx\, dy$$

Substituting in our reformulation of $p_n(x)$ as well as expanding $(1-y)^n$ allows us to employ lemma 5.1. It follows that

$$|I_n| = \left|\beta_n \zeta(2) - \frac{\alpha_n}{d_n^2}\right|$$

where α_n is an integer and

$$\beta_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{k}\frac{(n+k)!}{n!k!}.$$

It is clear that $|I_n| > 0$ since the integrand is positive for all $x, y \in (0, 1)$, to find an upper bound we substitute in $p_n(x)$ and perform integration by parts n times with respect to x. This yields

$$I_n = (-1)^n \int_0^1 \int_0^1 \frac{y^n(1-y)^n x^n(1-x)^n}{(1-xy)^{n+1}}\, dx\, dy.$$

To estimate this integral we need to consider the maximum value of the function

$$f(x, y) = \frac{x(1-x)y(1-y)}{1-xy}$$

By an application of the AM-GM inequality we see the following

$$\frac{x(1-x)y(1-y)}{1-xy} = \frac{xy(1-(x+y)+xy)}{1-xy} \leq \frac{xy(1-\sqrt{xy}+xy)}{1-xy} = \frac{xy(1-\sqrt{xy})}{1+\sqrt{xy}}.$$

Where equality occurs when $x+y = 2\sqrt{xy}$, ie $x = y$. Using the substitution $t = \sqrt{xy}$ for $t \in [0,1]$ we see that the function $f(t) = \frac{t^2(1-t)}{1+t}$ achieves its maximum at $t = \frac{-1+\sqrt{5}}{2}$ using ordinary calculus. It follows that

$$\frac{xy(1-y)(1-x)}{1-xy} \leq \left(\frac{\sqrt{5}-1}{2}\right)^5.$$

We can now estimate $|I_n|$ as follows

$$0 < |I_n| \leq \left|\int_0^1\int_0^1 \frac{y^n(1-y)^n x^n(1-x)^n}{(1-xy)^{n+1}}\,dx\,dy\right| \leq \left|\left(\frac{\sqrt{5}-1}{2}\right)^{5n}\int_0^1\int_0^1 \frac{1}{1-xy}\,dx\,dy\right| = \left(\frac{\sqrt{5}-1}{2}\right)^{5n}\zeta(2).$$

We also require some information on the size of d_n and β_n. Note that d_n is the product of all prime $p \leq n$ raised to the k such that $p^k \leq n$. That is,

$$d_n \leq \prod_{p\leq n} p^{\frac{\log(n)}{\log(p)}} = n^{\pi(n)}.$$

Where $\pi(n)$ is the prime counting function. It follows from the prime number theorem that for all $A > 1$ we must have $n^{\pi(n)} < e^{An}$. This give us

$$d_n \leq n^{\pi(n)} < e^{An}.$$

To analyse β_n we first recall that $n! \approx (\frac{n}{e})^n$, which comes from a weak form of Stirling's approximation. We see that for large n we have

$$\binom{n}{k}\binom{n}{k}\frac{(n+k)!}{n!k!} = \frac{n!(n+k)!}{(k!)^3((n-k)!)^2} \approx n^n(n+k)^{n+k}k^{-3k}(n-k)^{2k-2n}.$$

Where the powers of e sum to 0 and so cancel out. We wish to find the value of k that maximises this, so introduce the scaling $k = \alpha n$. This yields

$$\left(n(n+\alpha n)^{1+\alpha}(\alpha n)^{-3\alpha}(n-\alpha n)^{2\alpha-2}\right)^n = \left((1+\alpha)^{1+\alpha}\alpha^{-3\alpha}(1-\alpha)^{2\alpha-2}\right)^n.$$

Where again the sum of the powers of n is 0 and so cancel out. It remains to find the maximum value of

$$f(\alpha) = (1+\alpha)^{1+\alpha}\alpha^{-3\alpha}(1-\alpha)^{2\alpha-2}$$

for $0 < \alpha < 1$. The derivative can be computed and we find that we achieve the maximum at $\alpha = \frac{\sqrt{5}-1}{2}$. We can then simplify the maximum as

$$\begin{aligned} f\left(\frac{\sqrt{5}-1}{2}\right) &= \left(\frac{1+\sqrt{5}}{2}\right)^{\frac{1}{2}(1+\sqrt{5})}\left(\frac{\sqrt{5}-1}{2}\right)^{-\frac{3}{2}(\sqrt{5}-1)}\left(\frac{3-\sqrt{5}}{2}\right)^{\sqrt{5}-3} \\ &= \left(\frac{1+\sqrt{5}}{2}\right)^{\frac{1}{2}(1+\sqrt{5})}\left(\frac{1+\sqrt{5}}{2}\right)^{\frac{3}{2}(\sqrt{5}-1)}\left(\frac{1+\sqrt{5}}{2}\right)^{-2(\sqrt{5}-3)} \\ &= \left(\frac{1+\sqrt{5}}{2}\right)^5. \end{aligned}$$

We reach the bounds

$$c_1 n^{c_2}\left(\frac{1+\sqrt{5}}{2}\right)^{5n} \leq \beta_n \leq c_3 n^{c_4}\left(\frac{1+\sqrt{5}}{2}\right)^{5n}.$$

The constants can be computed explicitly by using a stronger form of Stirling's approximation, but we will not need this here. Setting $\gamma_n = d_n^2\beta_n$ and using the estimate for $|I_n|$ gives us

$$0 < \left|\zeta(2) - \frac{\alpha_n}{\gamma_n}\right| \leq \left(\frac{\sqrt{5}-1}{2}\right)^{5n}\frac{\zeta(2)}{\beta_n}.$$

We want our upper bound to be of the form set out in the definition of irrationality measure, we see that for sufficiently large n we have

$$\frac{1}{\gamma_n^{1+\delta}} = \left(\frac{\sqrt{5}-1}{1+\sqrt{5}}\right)^{5n}.$$

Taking logs on both sides and using the estimate for d_n yields

$$1+\delta = \frac{n\log[(1+\sqrt{5})/(\sqrt{5}-1)]^5}{\log(\beta_n d_n^2)} = \frac{\log[(1+\sqrt{5})/(\sqrt{5}-1)]^5}{\log\{[(1+\sqrt{5})/2]^5 e^2\}} = 1.092156\cdots > 1.$$

This concludes the proof since we have found an infinite sequence of rationals $\frac{\alpha_n}{\gamma_n}$ that approximate $\zeta(2)$ too well for it to be rational. □

We note that this is not the fastest or cleanest way of proving the result, in fact, it is not necessary to go through most of the estimates in the latter half of the proof. Showing that the upper bound on the integral goes to zero alongside the bound for d_n is enough for irrationality, see Beukers (1979).

The benefit of the proof shown above is that through the estimates we have explicitly constructed a sequence that estimates $\zeta(2)$, we also know exactly what the value of δ is. We can use this sequence to show that other rational approximations cannot do better than a bound that depends on δ. Informally, although the γ_n grow exponentially, the sequence is dense enough such that we can always choose one close enough to whatever q we choose.

Lemma 5.3. *Suppose there exists a sequence of rationals $\{p_n/q_n\}$ and $\delta > 0$ so that*

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n^{1+\delta}}$$

and the q_n satisfy the growth rate

$$q_n < q_{n+1} < q_n^{1+o(1)}.$$

Then either for some n we have

$$\frac{p}{q} = \frac{p_n}{q_n},$$

or for $\varepsilon > 0$ and a sufficiently large q

$$\left|\alpha - \frac{p}{q}\right| > \frac{1}{q^{1+1/\delta+\varepsilon}}.$$

Proof. This result can be found as an exercise in (Borwein & Borwein 1987, p. 376). We choose n such that $\frac{1}{2}q_{n-1}^{\delta} \leq q < \frac{1}{2}q_n^{\delta}$. For some $\varepsilon' > 0$ we let

$$\left|\alpha - \frac{p}{q}\right| = \frac{1}{q^{1+1/\delta+\varepsilon'}}.$$

Suppose that $p/q \neq p_n/q_n$, then by the triangle inequality we see

$$\frac{1}{qq_n} \leq \left|\frac{p}{q} - \frac{p_n}{q_n}\right| < \frac{1}{q_n^{1+\delta}} + \frac{1}{q^{1+1/\delta+\varepsilon'}}.$$

Thus

$$1 < \frac{q}{q_n^{\delta}} + \frac{q_n}{1/\delta + \varepsilon'}.$$

We now use the bounds chosen at the start of the proof to see that

$$\frac{q}{q_n^{\delta}} + \frac{q_n}{1/\delta + \varepsilon'} < \frac{1}{2} + \frac{q_n}{(\frac{1}{2})^{1/\delta+\varepsilon'} q_{n-1}^{1+\delta\varepsilon'}}.$$

This yields a contradiction due to the growth rate restriction, the right-hand side is less than 1 for large q. □

We see that $1 + \frac{1}{\delta} = 11.85078\ldots$

Theorem 5.4. *For p and q integers and q sufficiently large,*

$$\left|\pi^2 - \frac{p}{q}\right| > \frac{1}{q^{11.86}}.$$

Additionally,

$$\left|\pi - \frac{p}{q}\right| > \frac{1}{q^{23.72}}.$$

Proof. By the previous lemma it can be deduced that, given $\varepsilon > 0$, if

$$\left|\frac{\pi^2}{6} - \frac{p}{q}\right| < \frac{1}{q^{1+1/\delta+\varepsilon}}$$

for sufficiently large q, then we must have $p/q = \alpha_n/\gamma_n$. So it suffices to check the inequality for α_n/γ_n.

Note that for small $\nu > 0$ and large n we have

$$I_n > \left(\frac{\sqrt{5}-1}{2} - \nu\right)^{5n}.$$

It follows that

$$\left|\zeta(2) - \frac{\alpha_n}{\gamma_n}\right| \geq \frac{[(\sqrt{5}-1)/2-\nu]^{5n}}{\beta_n} \geq \frac{[(\sqrt{5}-1)/2-\nu]^{5n}}{\gamma_n} \geq \frac{1}{\gamma_n^3} > \frac{1}{\gamma_n^{11.86}}.$$

Where the penultimate inequality can be checked by multiplying both sides by β_n^2. The result follows by scaling p/q by 6. The irrationality measure for π follows immediately since for large q,

$$\left|\pi - \frac{p}{q}\right| = \frac{1}{|\pi + p/q|}\left|\pi^2 - \frac{p^2}{q^2}\right| > \frac{1}{|\pi + p/q|}\frac{1}{q^{2\times 11.86}}.$$

□

Variations of the technique shown here can be used to give upper bounds for the irrationality measures for other constants such as $\zeta(3)$ and $\log 2$. However none get close to 2 which is what we would expect from lemma 3.6. The issue with the approach here is the presence of d_n^2. Note that if we were able to remove this term[4] we would have $\delta = 1$, so the irrationality measure could be explicitly found using the constructed sequence.

The upper bound for π has been improved upon using more sophisticated versions of the argument exhibited here, but a precise value is currently out of reach. The table below gives some historical bounds for $\mu(\pi)$.

Upper Bound for $\mu(\pi)$	Reference	Year
$\mu(\pi) \leq 42$	Mahler	1953
$\mu(\pi) \leq 20.6$	Mignotte	1974
$\mu(\pi) \leq 14.65$	Chudnovsky	1982
$\mu(\pi) \leq 8.016\ldots$	Hata	1993
$\mu(\pi) \leq 7.606\ldots$	Salikhov	2008
$\mu(\pi) \leq 7.103\ldots$	Zudilin	2020

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[4]Calculating $1+\delta$ on page 20 without the e^2 term yields 2

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