Riemann Surfaces

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1 Introduction

In complex analysis, we studied the nature of holomorphic functions and the impacts of complex differentiability: these special functions introduce a plethora of unique attributes that we would never see in real analysis. These unique features lead into many profound theorems on $\mathbb{C}$, such as the identity theorem and open mapping theorem. But is $\mathbb{C}$ the only space that has functions with these powerful properties? In fact, $\mathbb{C}$ is just one of many spaces that facilitate holomorphic mappings: we call these spaces Riemann surfaces, named after the famous complex analyst Georg Friedrich Bernhard Riemann. Riemann surfaces are topological spaces, endowed with a structure that allows us to perform complex analysis on them.

Riemann surfaces are important for many reasons. On $\mathbb{C}$, we can encounter meromorphic functions, a special class of functions that, while not holomorphic on $\mathbb{C}$, can be studied as holomorphic mappings when extended to Riemann surfaces. Additionally, we will be focusing on topological properties of Riemann surfaces (and therefore spaces homeomorphic to them) using complex analysis techniques which would not be possible without the special structure that Riemann surfaces have. In particular, the Riemann-Hurwitz formula, which will be one of the main targets of our study, allows us to find out about the genus of compact Riemann surfaces by investigating holomorphic mappings on them.

Alongside some examples of Riemann surfaces like the Riemann sphere and complex torus, we will explore a variety of properties of holomorphic mappings including extensions of theorems we have encountered in complex analysis, such as the concept of holomorphic continuation. In addition, we will explore the nature of meromorphic functions and their links to Riemann surfaces. On the path to studying the Riemann-Hurwitz formula, we will also encounter the degree theorem, which uses the concept of valency to explore inconsistencies in holomorphic mappings. Finally, we will investigate some examples of the Riemann-Hurwitz formula in action, seeing how it interacts with meromorphic functions and eventually applying it to automorphisms of Riemann surfaces, leading to the discovery of Hurwitz surfaces, which allows us to connect complex analysis, topology, geometry and group theory in an exciting fashion.

I have also provided some case studies in the appendix to discuss additional interesting results of Riemann surfaces like the Weierstrass $\wp$ function and projective curves.

2 What are Riemann Surfaces?

Riemann surfaces were first defined in 1851 by Georg Friedrich Bernhard Riemann, during the completion of his PHD dissertation, where he introduced surfaces that could handle complex analysis [16, p.4].

2.1 Charts and Atlases

To begin, let's formalise the notion of a surface.

Definition 2.1 [6, p.29]

A surface R is a Hausdorff and 2nd countable topological space such that there exists an atlas (family of maps)

$$\{(\phi_\alpha, U_\alpha)\}_\alpha$$

where the coordinate function $\phi_\alpha : U_\alpha \to V_\alpha$ is a homeomorphism, $U_\alpha \subseteq R$ and $V_\alpha \subseteq \mathbb{C}$ are open sets, and $\{U_\alpha\}_\alpha$ is a cover of R. The pair (ϕ_α, U_α) is called a chart.

A surface can be seen as a shape that is covered by a set of maps. The earth can be imagined as a surface: it is not smooth or flat. There are hills and mountains, canyons and cliff sides. But we can break down the earth into tiny areas, like a single room or a small field. These little areas are flat, or at least could easily be flattened out. We can then represent this flattened area with a 2 dimensional plane, such as $\mathbb{C}$. This is the purpose of charts: they break down the surface into pieces small enough to be flattened and map them to complex planes, via a coordinate function. In other words, we are zooming into a specific region of a surface and describing it with complex numbers.

Definition 2.2 [6, p.29]

Let R be a surface with the atlas $\{(\phi_\alpha, U_\alpha)\}_\alpha$. For any pair of charts (U_α, ϕ_α) and (U_β, ϕ_β),

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$$

is a transition function.

An alternate way to look at a surface is as a series of complex planes (or subsets of complex planes) being stitched together. It is easy to interpret each complex plane when they are isolated, but on overlaps between these complex planes, we need to be able to traverse between them. Transition functions tell us how to move from one of these complex planes to another so we can traverse the surface in its entirety. For example, if we are on the ground (a flat area) and want to know how to reach the top of a nearby building (another flat area), a transition function would tell us to climb a ladder or staircase, providing a continuous method to move between the regions. We only care about transition functions that are defined on non-empty sets; if $U_\alpha \cap U_\beta = \emptyset$ then there is no point in investigating it.

Definition 2.3 [6, p.29]

Let R be a connected surface with the atlas $\{(\phi_\alpha, U_\alpha)\}_\alpha$. If all possible transition functions for this atlas are holomorphic functions, then $\{(\phi_\alpha, U_\alpha)\}_\alpha$ is a holomorphic atlas and R is a Riemann surface.

Transition functions are just functions from one subset of the complex plane to another so being holomorphic functions is well defined for them. This means that traversing through a Riemann surface is a smooth process: when seeing visualisations of Riemann surfaces they will appear somewhat polished, which is not necessary for regular surfaces. Recall from *MA3B8 Complex Analysis* that holomorphic functions preserves angles at which curves meet. This means that a Riemann surface locally mimics the geometric structure of a complex plane. Since each transition function's inverse is a transition function itself, all transition functions are biholomorphic functions.

2.2 Complex Structure

The coordinate function is a homeomorphism, making it bijective: there is only one element in the chart that maps to each element in the selected subset of the complex plane.

Definition 2.4 [8, p.10]

Let R be a Riemann surface with the holomorphic atlas $\{(\phi_\alpha, U_\alpha)\}_\alpha$. The chart (U_α, ϕ_α) is centred at $w \in R$ if $\phi_\alpha(w) = 0$.

So far, we have endowed surfaces with a single atlas and focused on that. However, there are many compatible atlases that cover surfaces. Furthermore, there are many holomorphic atlases that can cover Riemann surfaces. If we want a chart centred at a specific point, knowing an already established holomorphic atlas, we can find one by locating a chart containing that point in the holomorphic atlas and adjusting it slightly, for example, sometimes we can simply add a constant to the coordinate function. This will produce a new holomorphic atlas, but it will still preserve the structure of the Riemann surface while containing a chart centred at our desired point.

Definition 2.5 [15, p.18]

Let R be a Riemann surface with the holomorphic atlases $\{(\phi_\alpha, U_\alpha)\}_\alpha$ and $\{(\phi_\beta, U_\beta)\}_\beta$. These holomorphic atlases are equivalent if $\{(\phi_\alpha, U_\alpha)\}_\alpha \cup \{(\phi_\beta, U_\beta)\}_\beta$ is a holomorphic atlas.

This equivalence relation allows us to discover alternate holomorphic atlases once we have established a single holomorphic atlas. The equivalence class of this relation gives us many possible holomorphic atlases to choose from while maintaining the overall structure of the Riemann surface.

Definition 2.6 (The Complex Structure) [15, p.18]

Let R be a Riemann surface with the holomorphic atlas $\{(\phi_\alpha, U_\alpha)\}_\alpha$. The complex structure of R is the equivalence class of holomorphic atlases related to $\{(\phi_\alpha, U_\alpha)\}_\alpha$ under the equivalence relation above.

The benefit of focusing on a single holomorphic atlas is for convenience: we can establish Riemann surfaces by determining a holomorphic atlas, and it provides a concrete way to explore the Riemann surface. On the other hand, analysing the induced complex structure of the Riemann surface from this holomorphic atlas allows us to observe further properties. A single explicit choice of holomorphic atlas can limit our exploration of Riemann surfaces: some theorems may be difficult to apply to certain holomorphic atlases.

A Riemann surface's complex structure allows us to investigate global features of Riemann surfaces that are not dependent on a single, specific holomorphic atlas. For example, later on we will cover valency-related statements, which require every point in the Riemann surface to have a chart centred at it: if we were restricted to a single, explicit atlas, this would be very difficult! But from the complex structure, we can choose charts with the properties we desire. While the complex structure is induced by a holomorphic atlas, it is a more intrinsic property of the Riemann surface: once we have established the complex structure, we can use the charts within it without specifying the inductive holomorphic atlas.

2.3 The Complex Plane

Now let's consider some examples of Riemann surfaces. Of course, the complex plane and its connected subsets are trivial examples of Riemann surfaces.

Theorem 2.7 [3, p.4]

Let $\Omega \subseteq \mathbb{C}$ be open and connected. Then Ω is a Riemann surface.

Proof. Consider the coordinate function $\phi_1 : \Omega \to \Omega$, $\phi_1(z) = z$. Then $\{(\phi_1, \Omega)\}$ is an atlas. The only possible transition function $\phi_1 \circ \phi_1^{-1}(z) = z$ is a holomorphic function (as the identity function) so $\{(\phi_1, \Omega)\}$ is a holomorphic atlas and Ω is a Riemann surface. □

Even though this seems very obvious, explicitly defining this chart will give us some intuition when defining holomorphic functions on other, more complicated Riemann surfaces. We could centre the chart at different points by adding a constant to the coordinate function. The holomorphic atlas formed from this new translated chart is equivalent to the first one we formed, and so would be present in the complex structure induced by $\{(\phi_1, \Omega)\}$.

Does the complex structure contain every holomorphic atlas that covers a Riemann surface? No: let's consider a counterexample over Ω. Consider the atlas $\{(\phi_2, \Omega)\}$ given by $\phi_2 : \Omega \to \Omega'$, $\phi_2(z) = \overline{z}$, where Ω' contains the complex conjugates of all elements in Ω. Then $\phi_2 \circ \phi_2^{-1}(z) = z$ implies this is a holomorphic atlas ($\overline{z}$ is an involution so its inverse is itself), but $\phi_2 \circ \phi_1^{-1}(z) = \overline{z}$ is definitely not holomorphic, and so if both of these charts were in the same atlas, it would not be a holomorphic atlas. This example illustrates that it is possible for an induced complex structure to not contain every chart applicable for the Riemann surface.

2.4 The Riemann Sphere

In *MA3B8 Complex Analysis*, we explored the Riemann sphere, which we represent here by the extended complex plane. It turns out this space is in fact a Riemann surface.

Theorem 2.8 [5, p.3-4]

The Riemann sphere $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ is a compact Riemann surface.

The most useful holomorphic atlas for $\mathbb{C}_\infty$ for our purposes contains the following two charts: let the coordinate function $\phi_1 : \mathbb{C}_\infty \setminus \{0\} \to \mathbb{C}$ be

$$\phi_1(z) = \begin{cases} 0 & \text{if } z = \infty, \\ {}^1\!/_z & \text{otherwise,} \end{cases}$$

and let the coordinate function $\phi_2 : \mathbb{C} \to \mathbb{C}$ be $\phi_2(z) = z$. These coordinate functions are essentially stereographic projections from the Riemann sphere and as such are centred around ∞ and 0 respectively. It is clear the charts $(\phi_1, \mathbb{C}_\infty \setminus \{0\})$ and $(\phi_2, \mathbb{C})$ provide a cover for $\mathbb{C}_\infty$. On these charts, both coordinate functions are homeomorphisms. We can see that the intersection of the charts is

$$(\mathbb{C}_\infty \setminus \{0\}) \cap \mathbb{C} = \mathbb{C}^*,$$

where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Restricted to $\mathbb{C}^*$, ${}^1\!/_z$ is holomorphic. So $\phi_2 \circ \phi_1^{-1} = \phi_1^{-1}|_{\mathbb{C}^*}$ and $\phi_1 \circ \phi_2^{-1} = \phi_1|_{\mathbb{C}^*}$ are both holomorphic on $\mathbb{C}^*$, giving us a holomorphic atlas. Therefore, $\mathbb{C}_\infty$ is a Riemann surface.

The reason we can view $\mathbb{C}_\infty$ as a sphere is because there exists a homeomorphism between $\mathbb{C}_\infty$ and the unit sphere $\partial\mathbb{S}^2$, which we explored in *MA3B8 Complex Analysis*. As $\partial\mathbb{S}^2$ is a closed and bounded subset of $\mathbb{R}^3$, by the Heine-Borel theorem, it is compact. Therefore, the Riemann sphere, being topologically equivalent to $\partial\mathbb{S}^2$, is compact.

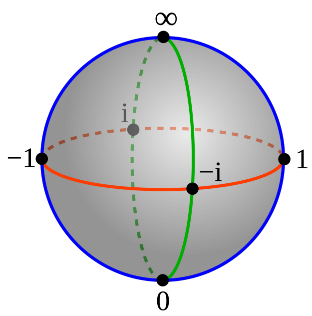

Figure 1: $\mathbb{C}_\infty$, the Riemann sphere [9]

What if we wanted a holomorphic atlas that was centred around different points instead of ∞ and 0? An example of a different atlas present in the complex structure induced by our earlier atlas is

$$\{(\phi_3, \mathbb{C}_\infty \setminus \{1\}), (\phi_4, \mathbb{C}_\infty \setminus \{-1\})\},$$

with the coordinate functions $\phi_3 : \mathbb{C}_\infty \setminus \{1\} \to \mathbb{C}$ as a stereographic projection from 1 and $\phi_4 : \mathbb{C}_\infty \setminus \{-1\} \to \mathbb{C}$ as a stereographic projection from -1. These new charts $(\phi_3, \mathbb{C}_\infty \setminus \{1\})$ and $(\phi_4, \mathbb{C}_\infty \setminus \{-1\})$ would be centred around -1 and 1 respectively.

2.5 The Complex Torus

The first research about the complex torus was completed by Carl Gustav Jacobi in 1834, but only when Riemann and Poincaré researched complex analysis further did it become clear how intricate tori could be [16, p.9-10]. Complex tori are constructed using a lattice.

Definition 2.9 [8, p.4]

A lattice Λ in $\mathbb{C}$ is an additive subgroup,

$$\Lambda = \{m\omega_1 + n\omega_2 \mid m, n \in \mathbb{Z}, \omega_1, \omega_2 \in \mathbb{C}^*\},$$

such that $\omega_1 \neq k\omega_2$ for $k \in \mathbb{R}$.

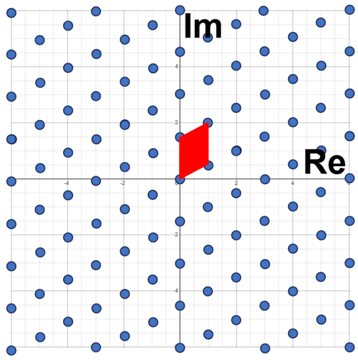

Figure 2: An example of a lattice using $3i$ and $1+2i$

In other words, the lattice points ω_1 and ω_2 would be linearly independent if considered as points in $\mathbb{R}^2$. Lattices contain points in the complex plane which, if connected by lines, would form a series of tessellating parallelograms.

Theorem 2.10 [8, p.4]

Let Λ be a lattice in $\mathbb{C}$. Then the quotient group $\mathbb{C}/\Lambda = \{z + \Lambda \mid z \in \mathbb{C}\}$ is a compact Riemann surface, known as the complex torus.

Recall from *MA3K4 Introduction to Group Theory* that this quotient group is the group of additive left cosets of the lattice. The torus is topologically a parallelogram with the edges identified, so we define the fundamental domain to be the parallelogram

$$\mathcal{P} = \{m\omega_1 + n\omega_2 \mid m, n \in [0,1) \text{ and } \omega_1, \omega_2 \in \mathbb{C}^*\}$$

which, in Figure 2, is represented in red. From the properties of cosets, we can observe that

$$\mathbb{C}/\Lambda = \{z + \Lambda \mid z \in \mathbb{C}\} = \{z + \Lambda \mid z \in \mathcal{P}\}.$$

To construct charts for an atlas describing this torus is, we consider a discrete set of complex numbers $\{w_\alpha\}_\alpha$. Let $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$ be the projection map. The coordinate function for each of these complex numbers is as follows:

$$\phi_\alpha : \pi(\mathbb{B}(w_\alpha, \varepsilon_\alpha)) \to \mathbb{B}(w_\alpha, \varepsilon_\alpha),$$

with ϕ_α being the inverse projection map and ε_α small enough that this π is injective in the chart. This ensures that the chart domains do not overlap themselves. Explicitly, the holomorphic atlas for the complex torus is $\{(\phi_\alpha, \pi(\mathbb{B}(w_\alpha))\}_\alpha$.

2.6 Compact Riemann Surfaces

Compactness plays a crucial role when exploring Riemann surfaces as it provides a greater deal of control over mappings between them. For example, within compact spaces, discrete sets must be finite, which we will later see leads to the concept of degrees of mappings between Riemann surfaces. Some examples of compact Riemann surfaces are:

- the Riemann sphere $\mathbb{C}_\infty$,
- the complex torus $\mathbb{C}/\Lambda$ for some lattice Λ,
- projective curves (see Appendix B).

By restricting ourselves to compact Riemann surfaces, we can find some interesting theorems: both the Riemann-Hurwitz formula and Hurwitz's automorphism theorem, which we will cover later, are fundamentally based on compact Riemann surfaces.

3 Holomorphic Mappings

We now take a look at functions and mappings between Riemann surfaces and explore how we create mappings with similar properties to holomorphic functions.

3.1 What are Holomorphic Mappings?

Holomorphic mappings are mappings which preserve the complex structure of Riemann surfaces.

Definition 3.1 [7, p.11]

Let R and S be Riemann surfaces and let $f : R \to S$ be a mapping. If there exists holomorphic atlases $\{(\phi_\alpha : U_\alpha \to V_\alpha, U_\alpha)\}_\alpha$ and $\{(\psi_\beta : W_\beta \to X_\beta, W_\beta)\}_\beta$ for R and S respectively, such that, for every pair of charts (ϕ_α, U_α) and (ψ_β, W_β),

$$\psi_\beta \circ f \circ \phi_\alpha^{-1} : V_\alpha \cap \phi_\alpha(f^{-1}(W_\beta)) \to X_\beta$$

is holomorphic, then f is a holomorphic mapping.

In other words, the function f is a holomorphic mapping if f is a holomorphic function (as seen in *MA3B8 Complex Analysis*) when composed with coordinate functions. The composition $\psi_\beta \circ f \circ \phi_\alpha^{-1}$ is called the local representation of the function on the complex plane and is an alternate way to view how points are mapped from one Riemann surface to another.

To illustrate this intuitively, we are looking at the behaviour of f throughout R and checking that it simulates a holomorphic function when considering how the relevant points act under coordinate functions. Specifically, we want to consider each $w \in R$ and its image $f(w) \in S$. We then observe whether the resulting function that maps $\phi_\alpha(w)$ to $\psi_\beta(f(w))$ is holomorphic for every chart that contains w and $f(w)$ (this is where the composition $\psi_\beta \circ f \circ \phi_\alpha^{-1}$ comes from). We repeat this for all points in R. Of course, this boils down to the simpler method of checking every relevant pair of charts.

Holomorphic mappings preserve the complex structure of each Riemann surface since they demand that, locally, the mapping must act like a holomorphic function. Since holomorphic functions are naturally preservative of angles and geometric structures, this behaviour is mirrored to holomorphic mappings, respecting the Riemann surface's complex structure.

Let's consider a basic example of a holomorphic mapping on a Riemann surface. If R is a Riemann surface with the holomorphic atlas $\{(\phi_\alpha : U_\alpha \to V_\alpha, U_\alpha)\}_\alpha$, let $\mathrm{Id} : R \to R$ be the identity mapping. We can check it is a holomorphic mapping by using

$$\phi_\beta \circ \mathrm{Id} \circ \phi_\alpha^{-1} = \phi_\beta \circ \phi_\alpha^{-1},$$

which we know is holomorphic as it is a transition function and on Riemann surfaces, transition functions are always holomorphic functions. Since this applies to every coordinate function, Id is a holomorphic mapping. Recall from *MA3K4 Introduction to Group Theory* that Id is actually an automorphism and is in fact the identity element of the automorphism group of R. In Section 7, we will explore the automorphism groups of Riemann surfaces in more detail.

3.2 Properties of Holomorphic Mappings

Many properties of holomorphic functions on $\mathbb{C}$ carry over to holomorphic mappings between Riemann surfaces.

Proposition 3.2

Let R and S be Riemann surfaces and $f : R \to S$ be a holomorphic mapping. Then f is continuous.

Proof. See Appendix D.1. □

It should be expected that holomorphic mappings are continuous. A less obvious property is holomorphic continuation, which let's us extend partially determined holomorphic mappings to larger portions of Riemann surfaces.

Theorem 3.3 (Holomorphic Continuation) [8, p.6-7]

Let R and S be Riemann surfaces and $f, g : R \to S$ be holomorphic mappings that coincide on $A \subset R$ with a limit point $a \in R$. Then $f \equiv g$.

So, if we know how a holomorphic mapping is defined on a subset of R, there is only one option to extend it throughout the entirety of R. The process of holomorphic continuation means that a holomorphic mapping defined on a single chart can be extended to other charts without disrupting the complex structure of the Riemann surface. This extends the identity theorem from *MA3B8 Complex Analysis* to Riemann surfaces. This is a special feature of complex analysis and holomorphic mappings which would not be observed with real differentiable functions.

Corollary 3.4 [10, p.6]

Let R and S be Riemann surfaces and $f : R \to S$ be a non-constant holomorphic mapping. For any $s \in S$, $f^{-1}(s)$ is a discrete set.

Proof. Let $f^{-1}(s)$ not be a discrete set, so there exists a sequence $w_n \to w \in f^{-1}(s)$ where w is an accumulation point. Let $g : R \to S$ be a constant holomorphic mapping with $g(z) = s$. Because $w \in f^{-1}(s)$, f and g coincide in some open neighbourhood of w with w as a limit point. So $f \equiv g$ by Theorem 3.3 and f is constant if $f^{-1}(s)$ is not a discrete set. □

This does not apply if f is constant as there are uncountably many pre-images in R for the constant.

Theorem 3.5 (Open Mapping Theorem) [8, p.10]

Let R be a Riemann surface and $f : R \to S$ be a non-constant holomorphic mapping. Then f is an open map.

Proof. See Appendix D.2. □

When we say f is an open map, we mean the image of an open set under f is always open. Here, we are extending the open mapping theorem that we saw in *MA3B8 Complex Analysis* to Riemann surfaces. The open mapping theorem essentially says that openness is maintained under holomorphic mappings. This is quite revealing of the nature of holomorphic mappings: they may be "stretchy" but cannot pierce or puncture the Riemann surface. Let's imagine open sets on Riemann surfaces with balloons. Filling a balloon with air will stretch and change its shape, but it will not tear the balloon (hopefully!) and similarly, non-constant holomorphic mappings may twist and turn open sets, but will not break them apart and create isolated points. Non-constant holomorphic mappings exist to transform shapes in a controlled manner, and a result of this is maintaining open sets. The same cannot be said about constant holomorphic mappings, as single points are closed sets on Riemann surfaces.

3.3 Meromorphic Functions

In the special case where the holomorphic mapping's codomain is $\mathbb{C}$, we call it a holomorphic function. They have all the properties that holomorphic mappings have, but we can define them in a simpler way:

Definition 3.6 [7, p.11]

Let R be a Riemann surface and let $f : R \to \mathbb{C}$ be a mapping. If there exists a holomorphic atlas $\{(\phi_\alpha : U_\alpha \to V_\alpha, U_\alpha)\}_\alpha$ for R, such that, for every chart (ϕ_α, U_α), $f \circ \phi_\alpha^{-1} : V_\alpha \to \mathbb{C}$ is holomorphic, then f is a holomorphic function.

This simpler definition arises from the fact that, as we saw in Theorem 2.7, $\mathbb{C}$ only requires one chart in its holomorphic atlas to cover it, for which the coordinate function can be the identity map. Therefore, in the context of Definition 3.1, ψ_β could be the identity map, simply leaving us with $f \circ \phi_\alpha^{-1}$, as seen in the definition above. However, as our focus later on will be on compact Riemann surfaces, holomorphic functions are not particularly useful to us:

Proposition 3.7 [8, p.11]

Let R be a compact Riemann surface and $f : R \to \mathbb{C}$ be a holomorphic function. Then f is constant.

Proof. Assume that f is not constant. By Theorem 3.5, f is an open map, meaning $f(R)$ is open. Because R is compact and f is continuous (by Proposition 3.2), $f(R)$ must be compact. Since $\mathbb{C}$ is a Hausdorff space, $f(R)$ must be closed - so $f(R)$ is both open and closed. Since $f(R) \subseteq \mathbb{C}$ and $\mathbb{C}$ is connected, $f(R) = \mathbb{C}$. But $\mathbb{C}$ is not compact which means $f(R) \neq \mathbb{C}$. So f cannot be non-constant.

To show a constant function does exist, let $f : R \to \mathbb{C}$, $f(z) = s$ be a constant function for $s \in \mathbb{C}$ and let $\{(\phi_\alpha : U_\alpha \to V_\alpha, U_\alpha)\}_\alpha$ be a holomorphic atlas for R. Then, from *MA3B8 Complex Analysis*, we know that $f \circ \phi_\alpha^{-1} : V_\alpha \to \mathbb{C}$ (which is still constant) is a holomorphic function. Therefore, a holomorphic function can exist on a compact Riemann surface but only if it is constant. □

If f is constant, then the open mapping theorem no longer applies, so $f(R)$ does not have to be open. Although $f(R)$ still needs to be closed by the argument above, this means $f(R)$ no longer is required to be $\mathbb{C}$, so f must be constant. As such, it is not possible to have any interesting holomorphic functions from a compact Riemann surface. This is not the case for general holomorphic mappings between two compact Riemann surfaces, which are a lot more flexible.

What happens if a function from a Riemann surface to $\mathbb{C}$ is not holomorphic everywhere? Normally, we would lose most of those incredibly useful properties we have established. However, there is a special kind of function which is still useful to us despite not necessarily being holomorphic everywhere: meromorphic functions.

Definition 3.8 [13, p.23]

Let R be a Riemann surface, $f : R \to \mathbb{C}$ be a mapping and $\Phi \subset R$ be a discrete set of points. If f is a holomorphic function on $R \setminus \Phi$ and, for all $w \in \Phi$, f has a pole at w, then f is a meromorphic function.

Away from a specific discrete set, meromorphic functions act exactly like holomorphic functions. However, as we traverse the Riemann surface and get closer to the special points within that discrete set, we diverge and travel far out in the complex plane: these points are called poles and a meromorphic function's value soars to ∞ as we approach them. Recall that the Riemann sphere contains ∞ as a point, so any sequences that would diverge on the complex plane will converge to ∞ on the Riemann sphere. That connects nicely to meromorphic functions: the only thing in the way of meromorphic functions being holomorphic is its discrete set of poles where they are undefined (but the function tend towards ∞). What would happen if we "filled in the dots" here using the Riemann sphere?

Lemma 3.9 [13, p.24]

Let R be a Riemann surface and $f : R \to \mathbb{C}$ be a meromorphic function. Then f can be extended to a holomorphic mapping $\tilde{f} : R \to \mathbb{C}_\infty$.

We can "fill in the dots" by setting $f(w) = \infty$ for all $w \in \Phi$ while extending $f : R \to \mathbb{C}$ to $\tilde{f} : R \to \mathbb{C}_\infty$. Now $\tilde{f}$ is a holomorphic mapping, spawned from a meromorphic function. Any properties of holomorphic mappings that meromorphic functions lacked are now accessible through this extension, such as continuity at every point in the Riemann surface R (which previously was not the case at the poles). Appendix A provides a case study of the Weierstrass $\wp$ function, a famous example of a meromorphic function.

3.4 Compact Maps

To explore further the properties of compact Riemann surfaces, we will look at compact maps.

Definition 3.10 [10, p.9]

Let X and Y be topological spaces and $f : X \to Y$ be a mapping. If, for all compact subsets $W_\beta \subset Y$, $f^{-1}(W_\beta) \subset X$ is compact, then f is a compact map.

As implied by its name, a compact map is a map which maintains compactness, similar to the way continuous maps work (with pre-images of open sets being open).

Proposition 3.11 [10, p.9]

Let R and S be compact Riemann surfaces and $f : R \to S$ be a holomorphic mapping. Then f is a compact map.

Proof. Any closed subset of a compact space is compact. Any compact subset of a Hausdorff space is closed. Therefore, any subset of a compact Hausdorff space is closed if and only if it is compact. Since R and S are compact Hausdorff spaces, this follows. As f is a holomorphic mapping, by Proposition 3.2, it is continuous, so the pre-image of any closed sets in S will be closed sets in R. Since all closed sets are compact and vice versa in a Hausdorff space, this means the pre-image of any compact set in S will be compact in R. So f is a compact map. $\square$

So, when using compact Riemann surfaces, we can guarantee that any holomorphic mapping is a compact map. This opens up some new possibilities: as a direct result, non-constant holomorphic mappings have finite pre-images.

Corollary 3.12

Let R and S be compact Riemann surfaces and $f : R \to S$ be a non-constant holomorphic mapping. Then $f^{-1}(s)$ is finite for any $s \in S$.

Proof. Since f is a holomorphic mapping on a compact Riemann surface, it is a compact map. Therefore, $f^{-1}(s)$ is compact. By Corollary 3.4, $f^{-1}(s)$ is also discrete, and a discrete compact set must be finite. $\square$

So the pre-image of any point in our codomain will only have a finite amount of elements: this is the foundation of the notion of degrees of holomorphic mappings. From now on, we make the general assumption that our mappings are never empty mappings (the image of the mapping is non-empty). Another feature of holomorphic mappings on compact Riemann surfaces is their surjectivity:

Proposition 3.13

Let R and S be compact Riemann surfaces and $f : R \to S$ be a non-constant holomorphic mapping. Then f is surjective.

Proof. Since f is open, $f(R)$ must be open by the open mapping theorem. Since the image of a compact set under a continuous map is compact, $f(R)$ must be compact and therefore closed. So $f(R)$ is both open and closed. Since S is connected, the only sets $f(R)$ could possibly be are S or $\emptyset$. By our assumption that f is defined, $f(R)$ must be S. So f is surjective. $\square$

Overall, holomorphic mappings are powerful mappings which maintain the complex structure of Riemann surfaces. When considering compact Riemann surfaces, holomorphic mappings offer a variety of further traits, like those discussed above. Because of these properties, we will see many theorems that apply exclusively to compact Riemann surfaces, such as the degree theorem.

4 Valency and Branch Points

Sometimes a holomorphic mapping does not behave normally or consistently at specific points: they may not be locally invertible or may experience a slower rate of change at these points. We label these points as valency points. Small changes from valency points lead to disproportionately smaller changes in the codomain of the mapping compared to other points.

4.1 What is Valency?

Recall that every holomorphic function on $\mathbb{C}$ is analytic: they can all be written in the form of a Taylor series.

Definition 4.1

Let $f : \Omega \to \mathbb{C}$ be a holomorphic function, where $\Omega \subseteq \mathbb{C}$, with the Taylor expansion at $w \in \Omega$,

$$f(z) = a_0 + a_1(z - w) + a_2(z - w)^2 + \dots$$

If $a_j = 0$ for all $j \geq 1$, the valency of w under f, $v_f(w) = \infty$. Otherwise $v_f(w) = j$ where $j > 0$ is the smallest j such that $a_j \neq 0$.

The valency of a point is linked to its behaviour in a small neighbourhood. The higher the valency of w, the more intensely f approaches a_0 in a local area of w. Recall that

$$a_j = \frac{f^{(j)}(w)}{j!},$$

so if the valency is ∞, then every derivative is 0 at that point, implying that the function is locally constant. This means that we are approaching a_0 with so much power that everything in its neighbourhood is a_0 itself: this notion leads to us labelling its valency ∞.

To illustrate valency, consider every point in the domain of the holomorphic function as an individual object with some mass. In reality, every object with mass exerts a gravitational force which creates a network of gravitational interactions connecting the objects. If we say that the valency at a point is directly proportional to the mass of its representation as an object, then points with higher valencies will exert a stronger gravitational attraction, as though the images of the points nearby to them were being pulled closer. A point with a higher valency have a stronger influence on a function's behaviour, similar to how objects with more mass have a stronger gravitational pull.

Valency can be similarly observed for holomorphic mappings, but we explore it via the local representation $\psi_\beta \circ f \circ \phi_\alpha^{-1}$, which we established in Definition 3.1, rather than using calculus.

Lemma 4.2 [8, p.10]

Let R and S be Riemann surfaces and $f : R \to S$ be a non-constant holomorphic mapping such that, for $w \in R$ and $s \in S$, $f(w) = s$. Then there exists charts (U_α, ϕ_α) and (W_β, ψ_β) in the complex structures of R and S centred around w and s respectively, such that $f(U_\alpha) \subset W_\beta$ and, for $n \in \mathbb{N}$,

$$\psi_\beta \circ f \circ \phi_\alpha^{-1} = z^n.$$

Essentially, the local representation can be of the form z^n if the charts are centred around the selected point and its image. This means that, outside of $w \in R$, every point in the codomain of the local representation will have n pre-images in the domain of the local representation. Using Lemma 4.2, we can now conceptualise valency for holomorphic mappings.

Definition 4.3 [8, p.10]

Let R and S be Riemann surfaces and $f : R \to S$ be a holomorphic mapping such that, for $w \in R$ and $s \in S$, $f(w) = s$. Let (U_α, ϕ_α) and (W_β, ψ_β) be charts in the complex structures of R and S centred around w and s respectively, such that $f(U_\alpha) \subset W_\beta$. Then the valency of w under f is

$$v_f(w) = \begin{cases} n & \text{if } \psi_\beta \circ f \circ \phi_\alpha^{-1} = z^n, \\ \infty & \text{if } \psi_\beta \circ f \circ \phi_\alpha^{-1} \text{ is constant,} \end{cases}$$

and w is a valency point if $v_f(w) > 1$. If w is a valency point, then $f(w)$ is a branch point.

Valency and branch points are essentially points which fail to be injective locally: they prevent the existence of a local inverse around these points. If a holomorphic mapping has no valency points, every point will have a local inverse.

Here, we have turned to the complex structure of the Riemann surface to choose (U_α, ϕ_α) and (W_β, ψ_β). We aim to select charts which are as small as possible to avoid the case of having two valency points in the same chart and so we are able to focus on a single point in particular. It is very easy to find a chart that is sufficient: we already know a chart containing w exists in the holomorphic atlas that induces the complex structure, so we can take a small subset of that and adjust the coordinate function so it is centred on w.

We define $\mathrm{Va}(f) = \{w \in R \mid v_f(w) > 1\}$ to be the valency locus and $\mathrm{Br}(f) = \{f(w) \in S \mid v_f(w) > 1\}$ to be the branch locus, which are the sets of valency and branch points respectively. While $\mathrm{Br}(f) = \mathrm{Im}(\mathrm{Va}(f))$, these two sets may have different cardinalities: multiple valency points can possibly map to the same branch point.

A special case is when the local representation $\psi_\beta \circ f \circ \phi_\alpha^{-1}$ is constant. The intuition here in labelling this as ∞ is due to the fact that, in the local representation, the infinite amount of points in the domain will be pre-images of a single point in the codomain.

Corollary 4.4

Let R and S be Riemann surfaces and $f : R \to S$ be a holomorphic mapping. If, for any $w \in R$, $v_f(w) = \infty$, then f is constant.

Proof. In Lemma 4.2, the sets in which f is being represented locally on are open sets. So the sets $\{w \mid v_f(w) = \infty\} \subset R$ and $\{w \mid v_f(w) < \infty\} \subset R$ are both open. The union of these sets is R, so these sets partition R. Because R is connected, it is impossible to split it into these two open sets unless one is empty, implying either $v_f(w) = \infty$ everywhere or $v_f(w) < \infty$ everywhere, and f will be constant if the former case holds. □

Corollary 4.4 means that, for a mapping to be holomorphic, it cannot go from a variable mapping to a constant mapping to a variable mapping again as you traverse the Riemann surface. Riemann surfaces being connected enforces this property. Since Corollary 4.4 shows that one point having valency ∞ means every point has valency ∞, we will ignore this case when dealing with non-constant mappings, instead focusing on finite valencies.

4.2 The Degree Theorem

Next we will investigate pre-images under holomorphic mappings between compact Riemann surfaces. To complicate things, for a holomorphic mapping f, the number of pre-images of a point depends on whether the point is in $\mathrm{Br}(f)$ and the value of its valency: there is not necessarily a uniform value for the number of pre-images any point has, unless $\mathrm{Va}(f) = \emptyset$.

We are interested in finding a classifier that can provide a "degree" for each non-constant holomorphic mapping which can give us some understanding of the mapping's pre-images. In fact, it turns out that, for any element in the codomain, if we add the valencies of all its pre-images, the sum is always the same.

Theorem 4.5 (Degree Theorem) [15, p.52]

Let R and S be compact Riemann surfaces and $f : R \to S$ be a non-constant holomorphic mapping. Let $\deg(f) : S \to \mathbb{N}$, the degree function, be

$$\deg(f)(s) = \sum_{\substack{w \in R \\ f(w)=s}} v_f(w).$$

Then $\deg(f)$ is constant.

We will refer to $\deg(f)(s)$ as just $\deg(f)$ since it is constant (unless it is helpful to specify s). As a result, we can observe that any point in $\mathrm{Br}(f)$ will have less pre-images than any point in $S \setminus \mathrm{Br}(f)$, as the degree of f will not differ at either of the points: they will have a matching total of valencies at their pre-images. If $s \notin \mathrm{Br}(f)$, then s always has $\deg(f)$ pre-images. The degree of a holomorphic mapping is always a positive number. This is due to valencies of points under holomorphic mappings being positive.

To illustrate the degree theorem, consider the meromorphic function $f : \mathbb{C}_\infty \to \mathbb{C}$ where $f(z) = {}^1\!/_{z^2}$, which has a pole at $z = 0$ and is undefined at $z = \infty$. Using Lemma 3.9, let's extend f to the holomorphic mapping $\tilde{f} : \mathbb{C}_\infty \to \mathbb{C}_\infty$ so $\tilde{f}(0) = \infty$ and $\tilde{f}(\infty) = 0$. Recall the coordinate functions $\phi_1 : \mathbb{C}_\infty \setminus \{0\} \to \mathbb{C}$ and $\phi_2 : \mathbb{C} \to \mathbb{C}$ from Theorem 2.8. Since $\phi_1(\infty) = 0$ and $\phi_2(0) = 0$, the charts are centred on 0 and ∞, so we can save time by focusing on just those points. We will calculate the valency at ∞. We look at the local representation

$$\phi_2 \circ \tilde{f} \circ \phi_1^{-1}(z) = \begin{cases} 0 & \text{if } z = 0, \\ {}^1\!/({}^1\!/_z)^2 & \text{otherwise,} \end{cases}$$

which simply evaluates to $\phi_2 \circ \tilde{f} \circ \phi_1^{-1}(z) = z^2$, lining up with our expectations from Lemma 4.2. This implies that $v_{\tilde{f}}(\infty) = 2$. Let's use the degree theorem (Theorem 4.5) at 0:

$$\deg(\tilde{f})(0) = \sum_{\substack{w \in \mathbb{C}_\infty \\ \tilde{f}(w)=0}} v_{\tilde{f}}(w) = v_{\tilde{f}}(\infty) = 2,$$

and because the degree is constant, $\deg(\tilde{f}) = 2$. The main takeaway is that we only need to calculate the degree at 0, and from there the degree theorem ensures the degree would have been the same value with a different point in $\mathbb{C}$. It also tells that the maximum valency any point can have is 2.

4.3 Covering Maps

Covering maps are ways of wrapping one space around another nicely. It you imagine a topological space as a big sheet, a covering map arranges this sheet neatly over another topological space. For example, the big sheet can be "folded" into smaller sheets, each of which is covering the codomain.

Definition 4.6 [6, p.46]

Let X and Y be topological spaces and $f : X \to Y$ be a mapping. Then, if for all $y \in Y$, there exists an open set $W \subset Y$ with $y \in W$ such that, for open sets $U_\alpha \subset X$,

$$f^{-1}(W) = \bigsqcup_{\alpha} U_\alpha,$$

and $f|_{U_\alpha}$ is a homeomorphism, f is a covering map.

If every point in the codomain of the covering map has the same number k of pre-images in the domain, then we called the covering map a k-sheeted covering map. By removing the valency locus and branch locus from compact Riemann surfaces, we are actually left with a covering map:

Proposition 4.7 [10, p.13]

Let R and S be compact Riemann surfaces and $f : R \to S$ be a non-constant holomorphic mapping. Let $S' = S \setminus \mathrm{Br}(f)$ and $R' = R \setminus \mathrm{Va}(f)$, so $f(R') = S'$. Then $f|_{R'} : R' \to S'$ is a $\deg(f)$-sheeted covering map.

Proof. See Appendix D.3. □

This result gives us some clarity on how to convert holomorphic mappings on Riemann surfaces with valency and branch points to covering maps and induces the main strategy we will use when proving the Riemann-Hurwitz formula.

5 Geometric and Topological Properties

Compact Riemann surfaces can be represented by contained geometric shapes, like we saw in Figure 1, with $\mathbb{C}_\infty$ being represented by a sphere, which are very helpful when trying to visualise them. When considering the geometric interpretation of a compact Riemann surface, there are visual properties of its shape which distinguish it from other Riemann surfaces and topological spaces.

5.1 Genera

One of the fundamental aspects of the shape of a compact Riemann surface is its genus.

Definition 5.1 [13, p.38]

Let R be a compact Riemann surface. The genus g_R of R is the maximum number of non-intersecting closed curves that can be placed on R without R becoming disconnected.

The genus is a topological property of compact surfaces. Riemann and Jordan discovered that the genus of a surface was consistent on all homeomorphic spaces and could be represented by a surface with a set number of handles [16, p.15]. As such, if we have a visualisation of a compact Riemann surface, the genus can be seen as the number of different places we could hold it from. For example, a sphere has no holes or gaps, implying that the Riemann sphere $\mathbb{C}_\infty$ has genus 0. On the other hand, a torus has genus 1.

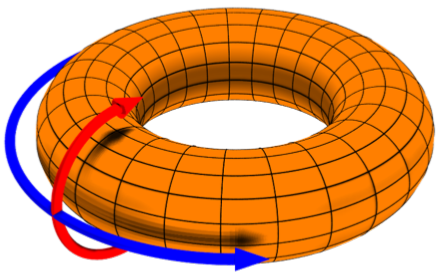

Figure 3: The complex torus [4]

As we can see in Figure 3, there is only 1 handle on the complex torus. This idea of handles relates to Definition 5.1 as we can view "holding" the surface as placing a closed curve on the surface - when we hold a mug, which is topologically equivalent to a torus, we place our fingers around the mug's handle, which forms a closed curve around the handle. The rest of the mug would still be connected, but placing another closed curve on it would make it disconnected topologically, implying that mugs have genus 1, like tori, which we would expect as it is a topological property.

5.2 Triangulations

When considering a geometric visualisation of a compact Riemann surface, one approach we can take is by overlapping our Riemann surface with a set of triangles.

Definition 5.2 [10, p.25]

Let R be a Riemann surface. A triangulation T on R is a set of triangle maps $\{t_\alpha : T_\alpha \to R\}_\alpha$, where $\{T_\alpha \subset \mathbb{C}\}_\alpha$ is a set of triangles that are either pairwise disjoint, intersect at a single vertex or intersect along edges, such that

$$R = \bigcup_{\alpha=1}^{n} t_\alpha(T_\alpha).$$

As we can see in Figure 4, a triangulation essentially tiles the Riemann surface with a tessellating pattern of triangles.

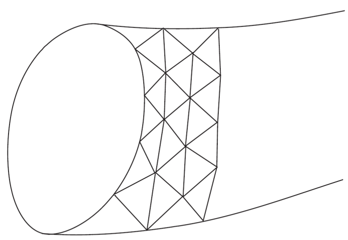

Figure 4: An example of a triangulation [6, p.98]

A triangulation forms a new object from these triangles: while this new object is not smooth like a Riemann surface, it is almost identical in its shape.

Proposition 5.3 [10, p.25]

Let R be a compact Riemann surface. A triangulation T of R contains only a finite amount of triangle maps.

Since we are focusing on compact Riemann surfaces, we do not have to worry about there being infinite triangles. An infinite number of triangles would imply that the surface is not bounded, violating the Riemann surface's compactness. This is similar to how we do not have to concern ourselves with non-constant holomorphic mappings having infinite cardinality pre-images. This would not necessarily be true for non-compact Riemann surfaces, which can be covered with an infinite amount of triangles.

5.3 The Euler Characteristic

The Euler characteristic is an alternative way of characterising the shape and structure of a compact Riemann surface, or topological spaces in general. We can calculate the Euler characteristic using a triangulation, since a triangulation of a compact Riemann surface is an almost identical shape to the Riemann surface itself.

Definition 5.4 [3, p.20]

Let R be a compact Riemann surface with a triangulation T containing $\mathcal{V}_T(R)$ vertices, $\mathcal{E}_T(R)$ edges and $\mathcal{F}_T(R)$ faces. The Euler characteristic of R is

$$\chi(R) = \mathcal{V}_T(R) + \mathcal{F}_T(R) - \mathcal{E}_T(R).$$

The actual value of $\chi(R)$ is the same regardless of the triangulation T we choose. To exploit this, we want a method to determine the Euler characteristic that does not depend on triangulations. To do this, notice that this formula is reminiscent of Euler's formula for polyhedra,

$$\mathcal{V} + \mathcal{F} - \mathcal{E} = 2.$$

Euler's formula only applies to shapes without holes in them. In fact, when a compact Riemann surface has genus 0, meaning it no holes, its Euler characteristic will always be 2. What happens with compact Riemann surfaces of higher genera? This will affect the Euler characteristic, as we will need more faces, edges and vertices to cover the Riemann surface. This overall change causes the Euler characteristic to actually decrease as we observe higher genera Riemann surfaces.

Lemma 5.5 [3, p.21]

Let R be a compact Riemann surface with genus g_R. Then the Euler characteristic of R is

$$\chi(R) = 2 - 2g_R.$$

As the Euler characteristic is independent of the triangulation, when choosing a triangulation for a Riemann surface, we can choose one with vertices at certain points and edges at selected lines.

6 The Riemann-Hurwitz Formula

The Riemann-Hurwitz formula, named after Georg Friedrich Bernhard Riemann and Adolf Hurwitz, brings together geometry, topology and complex analysis in regards to Riemann surfaces and holomorphic mappings.

Theorem 6.1 (Riemann-Hurwitz Formula) [10, p.25-26]

Let R and S be a compact Riemann surfaces and $f : R \to S$ be a non-constant holomorphic mapping. Let g_R and g_S be the genera of R and S respectively. Then

$$2g_R - 2 = \deg(f)(2g_S - 2) + \sum_{w \in R} (v_f(w) - 1).$$

Proof. We construct a triangulation T' of S such that the set of vertices of T' is a superset of the branch locus

$$\mathrm{Br}(f) = \{f(w) \mid w \in R, v_f(w) > 1\}.$$

Recall from Theorem 4.5 that, since $\deg(f)$ is constant, this set is finite. We focus on the pre-image of T' under f which we denote as T. By placing all the branch points as vertices, we can form a new map $f|_{R'} : R' \to S'$ with $S' = S \setminus \mathrm{Br}(f)$ and $R' = R \setminus \mathrm{Va}(f)$, the same restriction of f we saw in Proposition 4.7. By Proposition 4.7, $f|_{R'}$ is a $\deg(f)$-sheeted covering map implying that there will be $\deg(f)$ times as many faces and edges (since these are all contained in R' and S') within the triangulation T:

$$\mathcal{E}_T(R) = \deg(f)\mathcal{E}_{T'}(S)$$
$$\mathcal{F}_T(R) = \deg(f)\mathcal{F}_{T'}(S).$$

We can substitute these into the formula for the Euler characteristic of R with triangulation T:

$$\begin{aligned}\chi(R) &= \mathcal{V}_T(R) + \mathcal{F}_T(R) - \mathcal{E}_T(R) \\ &= \mathcal{V}_T(R) + \deg(f)\mathcal{F}_{T'}(S) - \deg(f)\mathcal{E}_{T'}(S).\end{aligned}$$

It may be natural to think that the number of vertices in T will also be $\deg(f)\mathcal{V}_{T'}(S)$, but that is not the case. We need to consider how many pre-images there are for vertices $s \in \mathrm{Br}(f)$. If $f^{-1}(s)$ contains valency point w, there will be $v_f(w) - 1$ less pre-images of s compared to if w was not a valency point, so we must subtract $v_f(w) - 1$ for every valency point in $f^{-1}(s)$ (we can further extend this to $w \in R$, for neater notation as any w that is not a valency point will have no impact on the summation). Taking this into account for all $s \in \mathrm{Br}(f)$, we can conclude that,

$$\mathcal{V}_T(R) = \deg(f)\mathcal{V}_{T'}(S) - \sum_{w\in R}(v_f(w) - 1).$$

To break this down: the first part, $\deg(f)\mathcal{V}_{T'}(S)$, shows the maximum amount of vertices in our triangulation, if there were no valency points. The second part,

$$-\sum_{w\in R}(v_f(w) - 1),$$

alerts us to vertices which are valency points, so every vertex represented in $\mathcal{V}_T(R)$ is unique. Using this result, we can observe

$$\begin{aligned}
\chi(R) &= \mathcal{V}_T(R) + \mathcal{F}_T(R) - \mathcal{E}_T(R) \\
&= \deg(f)\mathcal{V}_{T'}(S) - \sum_{w\in R}(v_f(w) - 1) + \deg(f)\mathcal{F}_{T'}(S) - \deg(f)\mathcal{E}_{T'}(S) \\
&= \deg(f)\chi(S) - \sum_{w\in R}(v_f(w) - 1).
\end{aligned}$$

Using Lemma 5.5, we can substitute the Euler characteristic for the genus, implying that

$$2g_R - 2 = \deg(f)(2g_S - 2) + \sum_{w\in R}(v_f(w) - 1),$$

which confirms the statement of the Riemann-Hurwitz formula. □

This means, assuming the requirements presented in Theorem 6.1 hold, if we know 3 of the following: the genus of the domain, the genus of the codomain, the degree of the mapping or the valency of points in that mapping, we can figure out at least some information about the fourth.

To show the Riemann-Hurwitz formula in action, let's apply it to meromorphic functions in the following simple situation. While $\mathbb{C}$ is not compact, $\mathbb{C}_\infty$ is a compact space, so extending meromorphic functions to holomorphic mappings in $\mathbb{C}_\infty$ will allow us to apply the Riemann-Hurwitz formula.

Corollary 6.2

Let R be a compact Riemann surface and $f : R \to \mathbb{C}$ be an injective meromorphic function. Then R has genus $g_R = 0$.

Proof. Extend f to the holomorphic mapping $\tilde{f} : R \to \mathbb{C}_\infty$: because f is injective, $\deg(\tilde{f}) = 1$ and $\tilde{f}$ is non-constant. Let's apply the Riemann-Hurwitz formula:

$$2g_R - 2 = \deg(\tilde{f})(2g_{\mathbb{C}_\infty} - 2) + \sum_{w\in R}(v_{\tilde{f}}(w) - 1).$$

By the degree theorem, Theorem 4.5, we know that, since $\deg(\tilde{f}) = 1$, the only possible value for $v_{\tilde{f}}(w)$ is 1 for all $w \in R$. As a result, there are no valency points, implying the summation evaluates to 0, leaving us with

$$2g_R - 2 = \deg(\tilde{f})(2g_{\mathbb{C}_\infty} - 2) = 2g_{\mathbb{C}_\infty} - 2.$$

We know that $g_{\mathbb{C}_\infty} = 0$, so we rearrange to see that $g_R = 0$. □

So we require meromorphic functions to be on genus 0 compact Riemann surfaces if they are injective. However, this does not apply for the converse: functions like $1/z^4$ (which has degree 4) can still exist on $\mathbb{C}_\infty$, but we would not see injective meromorphic functions from complex tori or other higher genera compact Riemann surfaces.

7 Hurwitz's Automorphism Theorem

Now that we have established the Riemann-Hurwitz formula, we can investigate automorphisms of Riemann surfaces. An automorphism on a compact Riemann surface R is a bijective holomorphic mapping, also known as a biholomorphic mapping, to itself, $\phi : R \to R$. The collection of automorphisms of R form the group $\mathrm{Aut}(R)$ under the composition action, using the identity automorphism $\mathrm{Id} : R \to R$ (which we considered earlier) as the identity element. When $g_R \geq 2$, $\mathrm{Aut}(R)$ contains finitely many automorphisms due to the restrictive shape of these spaces. This doesn't hold for $g_R = 0$ or 1 for which the Riemann surface can have infinitely many automorphisms, such as Möbius transformations for $\mathbb{C}_\infty$. Our aim is to find out the maximum number of automorphisms that can exist on R when $g_R \geq 2$, which we can use the Riemann-Hurwitz formula to deduce. From now on we refer to $\mathrm{Aut}(R)$ with A for simplicity.

Lemma 7.1

Let R be a compact Riemann surface with genus $g_R \geq 2$. Then the quotient surface R/A is a compact Riemann surface.

In R/A, we are identifying certain points with each other: we are taking $w \in R$, and identifying it with $\phi(w)$ for all $\phi \in A$. When you take the quotient of a Riemann surface, it still retains many of its original properties, including compactness. This is useful as we can create a holomorphic mapping between our original Riemann surface and the quotient surface.

Proposition 7.2 [1, p.22]

Let R be a compact Riemann surface with genus $g_R \geq 2$. Let $f : R \to R/A$ be the holomorphic mapping $f(z) = \{\phi(z) \mid \phi \in A\}$. Then $\deg(f) = |A|$.

To understand what is going on here, let's unpack f. For $w \in R$, $f(w)$ is applying every automorphism (remember that there is a finite number of automorphisms for $g_R \geq 2$) to w and sending all these outputs to a single element in R/A. This means that, if the element $w \in R$ is not a valency point, each automorphism acting on w sends it to a different point in R (except Id which sends it to itself). Therefore, the number of pre-images of $f(w)$ (a non-branch point) would be $|A|$, giving $\deg(f) = |A|$.

What about the branch points? The valency locus of f would be comprised of points in R with automorphisms that leave them fixed. We already know Id leaves all points in R fixed, so any other automorphisms which map a point w to itself will cause w to be a valency point (this also means there is one less pre-image for $f(w)$ which follows the degree theorem). Simply put, the valency points of f are points fixed by automorphisms other than Id.

Lemma 7.3 [1, p.23]

Let R be a compact Riemann surface with genus $g_R \geq 2$. Let $f : R \to R/A$ be the holomorphic mapping $f(z) = \{\phi(z) \mid \phi \in A\}$. If we treat A as a group action, then $v_f(w) = |\mathrm{Stab}_A(w)|$. Furthermore, the valency locus is

$$\mathrm{Va}(f) = \{w \mid |\mathrm{Stab}_A(w)| > 1\},$$

the set of points with non-trivial stabilisers under the group action A.

Here, we are considering A as a group action: for $w \in R$, $\mathrm{Orb}_A(w)$ is every point which an automorphism would map w to:

$$\mathrm{Orb}_A(w) = \{\phi(w) \mid \phi \in A\}.$$

$\mathrm{Stab}_A(w)$ is every automorphism which would map w to itself:

$$\mathrm{Stab}_A(w) = \{\phi \in A \mid \phi(w) = w\}.$$

We can equivalently think of f as collapsing each element in an orbit into 1 single element in R/A. From the degree theorem we can figure out the valency of each point: if the stabiliser of $w \in R$ has a higher order, clearly there are less pre-images for $f(w)$, implying the valency of each point increases with its stabiliser's cardinality.

Hurwitz's automorphism theorem, which Hurwitz published in the same paper that contained the Riemann-Hurwitz formula, gives us an upper bound for the order of automorphisms of a compact Riemann surface.

Theorem 7.4 (Hurwitz's Automorphism Theorem) [1, p.26-27]

Let R be a compact Riemann surface with genus $g_R \geq 2$. Then

$$|A| = |\mathrm{Aut}(R)| \leq 84(g_R - 1).$$

Proof. We can use the Riemann-Hurwitz formula and (for $f : R \to R/A$ where $f(z) = \{\phi(z) \mid \phi \in A\}$) apply $\deg(f) = |A|$ and Lemma 7.3 to see that

$$2g_R - 2 = |A|(2g_{R/A} - 2) + \sum_{w \in R} (|\mathrm{Stab}_A(w)| - 1).$$

For now, let's focus on the sum. We can view A as a group action, and the orbits are compressed into a single output: if there are n branch points, we can view them as the orbits

$$\mathrm{Orb}_A(w_1), \ldots, \mathrm{Orb}_A(w_n)$$

such that each of these orbits contain at least 1 element with a non-trivial stabiliser. Therefore,

$$\sum_{w \in R} (|\mathrm{Stab}_A(w)| - 1) = \sum_{j=1}^{n} \left(\sum_{w \in \mathrm{Orb}_A(w_j)} (|\mathrm{Stab}_A(w)| - 1) \right).$$

Recall that points in the same orbit will have conjugate stabilisers and so the valencies of points in the same orbit will be the same; this also implies that, if $w \in \mathrm{Orb}_A(w_j)$, then $|\mathrm{Stab}_A(w)| = |\mathrm{Stab}_A(w_j)|$. In addition, there are $|A|/|\mathrm{Stab}_A(w_j)|$ points in the orbit (by the orbit-stabiliser theorem from *MA3K4 Introduction to Group Theory*). Putting this all together, we observe that

$$\begin{aligned}
\sum_{j=1}^{n} \left(\sum_{w \in \mathrm{Orb}_A(w_j)} (|\mathrm{Stab}_A(w)| - 1) \right) &= \sum_{j=1}^{n} \frac{|A|}{|\mathrm{Stab}_A(w_j)|} (|\mathrm{Stab}_A(w_j)| - 1) \\
&= |A| \sum_{j=1}^{n} 1 - \frac{1}{|\mathrm{Stab}_A(w_j)|}.
\end{aligned}$$

Substituting this into the Riemann-Hurwitz formula gives

$$2g_R - 2 = |A| \left((2g_{R/A} - 2) + \sum_{j=1}^{n} 1 - \frac{1}{|\mathrm{Stab}_A(w_j)|} \right).$$

From here on, we want to maximise $|A|$. To do this, we will perform case analysis to minimise

$$(2g_{R/A} - 2) + \sum_{j=1}^{n} 1 - \frac{1}{|\mathrm{Stab}_A(w_j)|}.$$

The summation is always non-negative as $|\mathrm{Stab}_A(w_j)| \geq 1$. If $g_{R/A} \geq 2$, then $2g_R - 2 \geq 2|A|$, so $g_R - 1 \geq |A|$. Let's focus on the cases where $g_{R/A} = 1$ or 0 to see if we can achieve a higher bound. If $g_{R/A} = 1$, then

$$2g_R - 2 = |A| \sum_{j=1}^{n} 1 - \frac{1}{|\mathrm{Stab}_A(w_j)|}.$$

The smallest possible value for the sum is $1/2$, achieved when there is 1 branch point. This gives $4(g_R - 1) \geq |A|$. When $g_{R/A} = 0$, we are attempting to minimise

$$-2 + \sum_{j=1}^{n} 1 - \frac{1}{|\mathrm{Stab}_A(w_j)|}$$

to maximise $|A|$. For f to map a Riemann surface with genus 2 or more to a Riemann surface with genus 0, there must be at least 3 branch points (this can be seen from the Riemann-Hurwitz formula: since the left hand side is positive, the right hand side needs to be positive). Again, we will use n to refer to the number of branch points. If $n \geq 5$, then we can deduce

$$2g_R - 2 \geq |A| \left(-2 + \sum_{j=1}^{5} \frac{1}{2} \right)$$

which gives us the bound $4(g_R - 1) \geq |A|$. If $n = 4$, we cannot have all our stabiliser orders as 2, as $-2 + 4/2 = 0$. Instead, let one stabiliser have order 3, which gives us

$$2g_R - 2 \geq |A| \left(-2 + \frac{2}{3} + \sum_{j=1}^{3} \frac{1}{2} \right)$$

leading to a bound of $12(g_R - 1) \geq |A|$. Let's keep trying the other cases; the only option left is to have $n = 3$ branch points. Let

$$2 \leq |\mathrm{Stab}_A(w_1)| \leq |\mathrm{Stab}_A(w_2)| \leq |\mathrm{Stab}_A(w_3)|$$

without loss of generality (our lower bound is 2 otherwise it wouldn't be a valency point). If $|\mathrm{Stab}_A(w_1)| \geq 3$, the best choice is $|\mathrm{Stab}_A(w_1)| = 3$, $|\mathrm{Stab}_A(w_2)| = 3$ and $|\mathrm{Stab}_A(w_3)| = 4$ as these are the smallest values which don't cause our equation to evaluate to 0. Using these values, we achieve $24(g_R - 1) \geq |A|$. Now let's set $|\mathrm{Stab}_A(w_1)| = 2$. If $|\mathrm{Stab}_A(w_2)| = 4$, then we must choose $|\mathrm{Stab}_A(w_3)| = 5$ for the same reason as before. This combination leads to an upper bound of $40(g_R - 1) \geq |A|$. Now, $|\mathrm{Stab}_A(w_2)| \neq 2$ as we would get a negative number when evaluating, which is not possible from our earlier observations using the Riemann-Hurwitz formula. So $|\mathrm{Stab}_A(w_2)| = 3$, and we pick $|\mathrm{Stab}_A(w_3)| = 7$ as 3, 4, 5 and 6 would again have the same non-positive problem, so 7 is the smallest viable value we can choose. This is the last possible situation we can explore and it leads to the bound

$$84(g_R - 1) \geq |A|$$

achieving what is known as the Hurwitz bound. □

So, not only is the automorphism group of a compact Riemann surface finite when $g_R \geq 2$, but we can identify an upper bound dependent on the genus. However, most compact Riemann surfaces will not have automorphism groups with order $84(g_R - 1)$. For example, there is no compact Riemann surface with genus 2 that has an automorphism group containing 84 elements. In the special case that a compact Riemann surface does fulfill the Hurwitz bound property, we give the space a special name.

Definition 7.5 [14, p.1-7]

Let H be a compact Riemann surface with genus $g_H \geq 2$. H is a Hurwitz surface if

$$|\mathrm{Aut}(H)| = 84(g_H - 1).$$

The Klein quartic KQ, named after Felix Klein, is an example of a Hurwitz surface. It can be viewed as a projective curve (see Appendix B):

$$\mathrm{KQ} = \{[z_1 : z_2 : z_3] \in \mathbb{P}^2 \mid z_1^3 z_2 + z_2^3 z_3 + z_3^3 z_1 = 0\},$$

and has genus $g_{\mathrm{KQ}} = 3$. This would imply that the upper bound of $|\mathrm{Aut}(\mathrm{KQ})|$ is $84(3-1) = 168$. In *MA3B8 Complex Analysis*, we explored the PSL groups, and it turns out that Aut(KQ) is isomorphic to $PSL(2, \mathbb{F}_7)$, consisting of invertible 2-matrices, with elements from $\mathbb{F}_7$, with determinant 1. This is because we can view the Klein quartic as a tiling of regular heptagons, with 3 heptagons meeting at each vertex, as seen in Figure 5, and the symmetries of KQ (such as rotations and reflections) consistently maintain this structure: there are 168 of these possible symmetries, which are the automorphisms. Similarly, $PSL(2, \mathbb{F}_7)$ contains 168 elements, which can each correspond to a specific symmetry.

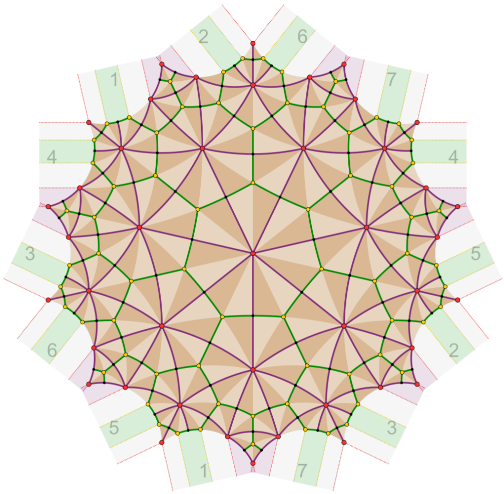

Figure 5: Tiling the Klein quartic [12]

In fact, KQ is the only Hurwitz surface with genus 3. As we mentioned earlier, there are no Hurwitz surfaces with genus 2, so KQ is the Hurwitz surface with the lowest genus. In fact there are no more Hurwitz surfaces until we reach genus 7, with the Macbeath curve, defined by the projective curve (see Appendix B),

$$\mathrm{MB} = \{[z_1 : z_2 : z_3] \in \mathbb{P}^2 \mid z_1^7 + z_2^7 + z_3^7 - k(z_1^3 z_2^3 z_3 + z_1^3 z_2 z_3^3 + z_1 z_2^3 z_3^3) = 0\},$$

where k is an element of the field in which MB is defined. Similar to Aut(KQ), Aut(MB) is isomorphic to $PSL(2, \mathbb{F}_8)$ with 504 symmetries, giving us 504 automorphisms in Aut(MB). Hurwitz surfaces are very rare and the fact that the order of $\mathrm{Aut}(H)$ is exactly $84(g_H - 1)$ and not just bounded by it reflects the incredibly symmetric nature of Hurwitz surfaces.

8 Conclusion

Our exploration of the Riemann-Hurwitz formula has provided us insight into Riemann surfaces and holomorphic mappings between them. By proving the Riemann-Hurwitz formula, we have enriched our understanding of the underlying geometry behind compact Riemann surfaces, especially with our focus on genera and how the valency degree of holomorphic mappings affects it. Furthermore, we used the Riemann-Hurwitz formula to prove Hurwitz's automorphism theorem, providing an upper bound to the automorphism groups of compact Riemann surfaces and building a connection between group theory, complex analysis, geometry and topology. In Appendices A, B and C, there are examples of the Riemann-Hurwitz formula in action with meromorphic functions and projective curves.

Looking ahead, there are several areas of research to expand further into. Firstly, the Riemann-Hurwitz formula can be used to prove the Riemann-Roch theorem, which uses differential forms to find the dimension of a field of meromorphic functions with known poles and zeros. Additionally, Hurwitz's automorphism theorem is instrumental in group theoretic research of Riemann surfaces and Hurwitz surfaces. In more applied settings, the Riemann-Hurwitz formula can be used to analyse the security of cryptographic systems in elliptic curve cryptography and has other uses throughout physics, particularly in string theory and quantum field theory.

A Case Study: The Weierstrass $\wp$ Function

Let's take a look at famous example of a meromorphic function: the Weierstrass $\wp$ function (also known as the Weierstrass elliptic function). This unique meromorphic function was named after Karl Weierstrass, but many of the discoveries regarding elliptic functions were found independently by Abel and Jacobi [16]. Before we apply Riemann surfaces to the problem, let's consider the function from $\mathbb{C}$ to $\mathbb{C}$. If Λ is a lattice in $\mathbb{C}$, the Weierstrass $\wp$ function $\wp : \mathbb{C} \to \mathbb{C}$ is

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right).$$

The function $\wp$ is holomorphic at any $w \in \mathbb{C} \setminus \Lambda$: when $w \in \Lambda$, it is a pole of $\wp$. Since Λ is a discrete set, $\wp$ is meromorphic. Unlike some of the more basic meromorphic functions on $\mathbb{C}$ (such as $1/z$), $\wp$ has an infinite number of poles: at every one of the infinite lattice points, there is a pole. This does not invalidate being a meromorphic function, as these points still form a discrete set, but we can actually adjust this function so that it has a finite set of poles.

For any $\lambda \in \Lambda$, $\wp(z+\lambda) = \wp(z)$ and so this function is doubly-periodic (when considered geometrically, there are two directions in which $\wp$ is periodic). Recall that $\mathbb{C}/\Lambda = \{z + \Lambda \mid z \in \mathcal{P}\}$ is the complex torus with respect to the lattice Λ, with $\mathcal{P}$ being this torus' fundamental domain (which is a parallelogram), which is in red in Figure 2. Because it is doubly-periodic, we can consider the following modification of $\wp$ on $\mathbb{C}/\Lambda$.

Proposition A.1 [2, p.85]

Let Λ be a lattice in $\mathbb{C}$ and $\wp : \mathbb{C}/\Lambda \to \mathbb{C}$ be

$$\wp(z+\Lambda) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right).$$

Then $\wp$ is a meromorphic function with finitely many poles.

The function being holomorphic (away from poles) is preserved as we travel through the torus, due to it being doubly-periodic: travelling along to the same position of each identified edge gives the same result, maintaining continuity throughout the complex torus. Since we have identified every $\lambda \in \Lambda$, there is just a single pole. We need to test the local representation

$$\wp \circ \phi_\alpha^{-1}$$

away from this pole to see if it is holomorphic; we can use the coordinate functions we described in Theorem 2.10. Since ϕ_α^{-1} is just a projection mapping onto the torus, and we know projection mappings are holomorphic, this is a holomorphic function away from lattice points. So $\wp$ is a meromorphic function on $\mathbb{C}/\Lambda$ with only one pole.

Finally, let's use Lemma 3.9 to extend $\wp$ to the holomorphic mapping $\tilde{\wp} : \mathbb{C}/\Lambda \to \mathbb{C}_\infty$, by letting the pole in $\wp$ map to ∞. The benefit to doing this is that both $\mathbb{C}/\Lambda$ and $\mathbb{C}_\infty$ are compact Riemann surfaces. Now, we can use the Riemann-Hurwitz formula to explore further properties of $\wp$.

Proposition A.2 [15, p.68-69]

Let Λ be a lattice in $\mathbb{C}$ and $\tilde{\wp} : \mathbb{C}/\Lambda \to \mathbb{C}_\infty$ be

$$\tilde{\wp}(z+\Lambda) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right).$$

Then $|\mathrm{Va}(\tilde{\wp})| = 4$, that is, there are 4 valency points.

Proof. We already established that $\wp$ only has one pole so $\tilde{\wp}$ will only output ∞ at one point: the lattice points are all identified with each other. Since this point has valency 2 (which is easily observed from the ${}^{1}/_{z^2}$ term and our example in Section 4.2), the degree of $\tilde{\wp}$ is 2. Let's see what is left of the Riemann-Hurwitz formula after setting $g_{\mathbb{C}_\infty} = 0$:

$$\begin{aligned}
2g_{\mathbb{C}/\Lambda} - 2 &= -2\deg(\tilde{\wp}) + \sum_{w\in\mathbb{C}/\Lambda} (v_{\tilde{\wp}}(w) - 1) \\
&= -2 \sum_{\substack{w\in\mathbb{C}/\Lambda \\ \tilde{\wp}(w)=\infty}} v_{\tilde{\wp}}(w) + \sum_{w\in\mathbb{C}/\Lambda} (v_{\tilde{\wp}}(w) - 1) \\
&= -2(2) + \sum_{w\in\mathbb{C}/\Lambda} (v_{\tilde{\wp}}(w) - 1).
\end{aligned}$$

Since the genus of a complex torus is 1, we know that $2g_{\mathbb{C}/\Lambda} - 2 = 0$. Therefore

$$\begin{aligned}
&-4 + \sum_{w\in\mathbb{C}/\Lambda} (v_{\tilde{\wp}}(w) - 1) = 0 \\
\implies &\sum_{w\in\mathbb{C}/\Lambda} (v_{\tilde{\wp}}(w) - 1) = 4
\end{aligned}$$

so there are at most 4 valency points for $\tilde{\wp}$. However, since we know that the degree is 2, each valency point can only have valency 2, meaning there are exactly 4 valency points. □

So where exactly are these 4 valency points? Recall that valency points interrupt the local invertibility of their neighbourhoods, so if we consider $\wp : \mathcal{P} \to \mathbb{C}$, where $\mathcal{P}$ is the fundamental domain, we can look at where $\wp'(z) = 0$ and use this to see where valency points would be induced on our mapping. The derivative of $\wp$ on $\mathbb{C}$ is

$$\wp'(z) = -\sum_{\lambda\in\Lambda} \frac{2}{(z-\lambda)^3},$$

which, as an odd doubly-periodic function, has zeros at ${}^{\omega_1}/_2$, ${}^{\omega_2}/_2$ and ${}^{\omega_1+\omega_2}/_2$, where ω_1 and ω_2 are the two lattice points of the lattice Λ. This induces three of the valency points on the mapping from $\mathbb{C}/\Lambda$. What about the fourth valency point? Since we are aware that $\wp'$ induces valency points, by the degree theorem, we know that if $\wp'$ has a point in its codomain with a single pre-image, this pre-image will also induce a valency point on the mapping from $\mathbb{C}/\Lambda$. Since 0 is the only point in $\mathcal{P}$ that maps to ∞, 0 must also induce a valency point. Therefore, the valency points of $\tilde{\wp} : \mathbb{C}/\Lambda \to \mathbb{C}_\infty$ are, in terms of cosets of $\mathbb{C}/\Lambda$,

$$\mathrm{Va}(\tilde{\wp}) = \left\{\Lambda, \frac{\omega_1}{2} + \Lambda, \frac{\omega_2}{2} + \Lambda, \frac{\omega_1+\omega_2}{2} + \Lambda\right\}.$$

B Case Study: Projective Curves

Using projective spaces, we can explore many more Riemann surfaces. The projective space is the space composed of complex lines, containing elements which are ratios of complex numbers.

Definition B.1 [6, p.33-34]

The projective space $\mathbb{P}^n$ is the space of complex lines in $\mathbb{C}^{n+1}$, where $[w_1 : w_2 : \cdots : w_{n+1}] \in \mathbb{P}^n$, with each $w_\alpha \in \mathbb{C}$, has the property that, for all $k \in \mathbb{C}^*$,

$$[w_1 : w_2 : \cdots : w_{n+1}] = [kw_1 : kw_2 : \cdots : kw_{n+1}].$$

Additionally, an element of $\mathbb{P}^n$ must have at least one non-zero w_α. Put differently, $[0 : \cdots : 0] \notin \mathbb{P}^n$: there's no line from the origin to itself. To construct $\mathbb{P}^n$, we use a quotient map on $\mathbb{C}^{n+1}$, $\mathbb{C}^{n+1}/(w \sim kw)$, which identifies points that are scalar multiples of one another. Therefore, the open sets in $\mathbb{P}^n$ are defined by the quotient topology: a set in $\mathbb{P}^n$ is open if its pre-image under the quotient map is open.

Proposition B.2

The projective space $\mathbb{P}^n$ is compact.

Proof. Let $[w_1 : w_2 : \cdots : w_{n+1}] \in \mathbb{P}^n$ and $k = ||(w_1, w_2, \ldots, w_{n+1})||$. Then

$$[w_1 : w_2 : \cdots : w_{n+1}] = [{}^{w_1}/_k : {}^{w_2}/_k : \cdots : {}^{w_{n+1}}/_k].$$

This implies the entirety of $\mathbb{P}^n$ can be viewed on the unit sphere (the boundary of the unit ball) of $\mathbb{C}^{n+1}$, which we will denote by $\partial\mathbb{S}^{2n+1}$. Intuitively, this makes sense, as $\mathbb{P}^n$ is the set of complex lines through the origin: these lines are guaranteed to pierce the unit sphere at 2 points. By the Heine-Borel theorem, $\partial\mathbb{S}^{2n+1}$ is compact as it is a closed and bounded subset of $\mathbb{C}^{n+1}$ (which is homeomorphic to $\mathbb{R}^{2n+2}$). Let $\pi : \partial\mathbb{S}^{2n+1} \to \mathbb{P}^n$ be the projection map. Then π is continuous: all pre-images of open sets in $\mathbb{P}^n$ are open sets in $\mathbb{C}^{n+1}$ as a result of using the quotient topology. In addition, π is surjective as we established earlier that $\mathbb{P}^n$ can entirely be viewed in the unit sphere. As π is continuous and surjective, and $\partial\mathbb{S}^{2n+1}$ is compact, $\mathbb{P}^n$ is compact. $\square$

A basic consequence of defining the projective space is that $\mathbb{P}^1$ is topologically equivalent to $\mathbb{C}_\infty$. To see this, consider that $f : \mathbb{P}^1 \to \mathbb{C}_\infty$, $f([z_1 : z_2]) = {}^{z_1}/_{z_2}$ is a homeomorphism. The space $\mathbb{P}^2$ is more interesting:

Theorem B.3 [11, p.16]

Let $P(z_1, z_2, z_3)$ be a non-singular (except possibly at $(0, 0, 0)$), non-constant, homogeneous polynomial. Then the projective curve based on the polynomial's zero locus,

$$\{[z_1 : z_2 : z_3] \in \mathbb{P}^2 \mid P(z_1, z_2, z_3) = 0\}$$

is a compact Riemann surface.

Some of the conditions here are quite specific. Homogeneity is needed since the scalar multiple property holds: if the polynomial is not homogeneous, multiplying each solution $[z_1 : z_2 : z_3]$ by a scalar $k \in \mathbb{C}^*$ (which is an operation that $[z_1 : z_2 : z_3]$ should stay the same under), may result in a different solution for the zero locus. This is not something that can happen when when the polynomial is homogeneous as we can simply factor the scalar out and return to the original terms.

By non-singular, we mean that at least one of the partial derivatives is non-zero at every point. We need non-singular polynomials since singularities can interrupt the smoothness of the generated surface. For example, a singularity could cause a cusp to appear on the surface. This would violate the condition that makes Riemann surfaces special: their smoothness. This is actually a consequence of the implicit function theorem, extended to complex variables: the implicit function theorem guarantees the existence of a holomorphic function in a small neighbourhood of non-singular points. This implication allows us to construct holomorphic atlases on the projective curve to give it a Riemann surface structure. We do not have to worry about $(0, 0, 0)$ being a singularity though, because $[0 : \cdots : 0]$ is not actually a point in $\mathbb{P}^n$.

As closed subsets of $\mathbb{P}^2$, projective curves are compact. Projective curves are closed as they are the pre-image of the closed set $\{0\}$ under the continuous polynomial map P. An example of a projective curve is the Klein quartic curve, which we discussed earlier:

$$\mathrm{KQ} = \{[z_1 : z_2 : z_3] \in \mathbb{P}^2 \mid z_1^3 z_2 + z_2^3 z_3 + z_3^3 z_1 = 0\},$$

which results in a Hurwitz surface. Another example of a projective curve is the Fermat curve of degree d,

$$F_d = \{[z_1 : z_2 : z_3] \in \mathbb{P}^2 \mid z_1^d + z_2^d = z_3^d\},$$

named after Fermat's last theorem - it contains complex solutions to the famous equation. We have a simple formula to calculate the genus of a Fermat curve.

Theorem B.4 [15, p.58-60]

Let F_d be the Fermat curve of degree d. Then the genus of F_d is

$$g_{F_d} = \frac{(d-1)(d-2)}{2}.$$

Proof. By looking at $P(z_1, z_2, z_3) = z_1^d + z_2^d - z_3^d$, it is clear that F_d extends to a compact Riemann surface. We consider the holomorphic mapping $f : F_d \to \mathbb{C}_\infty$,

$$f([z_1 : z_2 : z_3]) = \frac{z_1}{z_2}.$$

Let's first sort out the pre-image of ∞. Let $k \in \mathbb{C}^*$, then

$$f^{-1}(\infty) = f^{-1}\left(\frac{k}{0}\right) = \{[k : 0 : z_3] \mid k = z_3^d\},$$

which has no valency points and d elements: we can see that $\deg(f) = d$ (by constancy of the degree we only needed to look at the pre-image of one point). We already know that $g_{\mathbb{C}_\infty} = 0$ so all we need to do is find the valency locus. Every other point in $\mathbb{C}_\infty$ is a standard complex number. Consider the general pre-image of a point $z_1 \in \mathbb{C}$,

$$f^{-1}(z_1) = \{[z_1 : 1 : z_3] \mid z_1^d + 1 = z_3^d\}.$$

Then z_3 has d possible values except when $z_1^d = -1$, where z_3 only has one possible value of 0: we know from the constancy of the degree that these points must have valency d. This occurs d times, as there are d complex values that can result in $z_1^d = -1$. Now, let's put this all into the Riemann-Hurwitz formula:

$$\begin{aligned}
2g_{F_d} - 2 &= \deg(f)(2g_{\mathbb{C}_\infty} - 2) + \sum_{w \in R}(v_f(w) - 1) \\
&= d(0-2) + \sum_{w \in \mathrm{Va}(f)} (d-1) \\
&= -2d + d(d-1) \\
&= d^2 - 3d.
\end{aligned}$$

By rearranging the equation, we can observe that

$$\begin{aligned}
2g_{F_d} &= d^2 - 3d + 2 \\
\implies g_{F_d} &= \frac{(d-1)(d-2)}{2}
\end{aligned}$$

so we can deduce the genus of any degree Fermat curve. □

The compactness of projective curves allows us to apply the Riemann-Hurwitz formula, giving us a fast method to work out the genus of the Fermat curve.

C Case Study: Conformal Equivalence

Earlier, we saw an equivalence relation between holomorphic atlases on Riemann surfaces, leading to the idea of complex structure, but we will now look for an equivalence relation between Riemann surfaces that may appear to be different.

C.1 Biholomorphic Mappings

In Section 7, we covered biholomorphic mappings (bijective holomorphic mappings) from Riemann surfaces to themselves, which are automorphisms. On a wider scale, we can define a notion of conformal equivalence, like we saw in *MA3B8 Complex Analysis*, using possible biholomorphic mappings between Riemann surfaces.

Definition C.1 [8, p.6]

Let R and S be Riemann surfaces. If there exists a biholomorphic mapping $f : R \to S$, then R and S are conformally equivalent.

So when are two compact Riemann surfaces conformally equivalent? Earlier, we covered that $\mathbb{P}^1$ was homeomorphic to $\mathbb{C}_\infty$. However, $f([z_1 : z_2]) = {}^{z_1}/_{z_2}$ is not just a homeomorphism, but a biholomorphic mapping. Therefore, these spaces are conformally equivalent, sharing their complex structure. For other compact Riemann surfaces, from the Riemann-Hurwitz formula, we can see that the genus of both R and S need to be the same to guarantee the degree of the biholomorphic mapping is 1, otherwise it will not be bijective. Of course, this is reaffirmed by the fact that biholomorphic mappings are clearly always homeomorphisms, preserving topological properties. But maintaining topological properties is not enough to guarantee conformal equivalence: biholomorphic mappings have stronger conditions than homeomorphisms. Fortunately, when we consider compact Riemann surfaces, there is a way we can explore conformal equivalence: these Riemann surfaces have the special property of always being conformally equivalent to a projective curve in $\mathbb{P}^2$.

Proposition C.2 [6, p.50-52]

Let R be a compact Riemann surface. Then there exists a non-singular (except possibly at $(0, 0, 0)$), non-constant, homogeneous polynomial $P(z_1, z_2, z_3)$, such that R is conformally equivalent to

$$\{[z_1 : z_2 : z_3] \in \mathbb{P}^2 \mid P(z_1, z_2, z_3) = 0\}.$$

So if two compact Riemann surfaces are conformally equivalent to the same projective curve, they must be conformally equivalent to each other by transitivity. $\mathbb{C}_\infty$ is not just conformally equivalent to $\mathbb{P}^1$, but also has a biholomorphic mapping to $\{[z_1 : z_2 : z_3] \in \mathbb{P}^2 \mid z_3 = 0\}$, given by $f(z) = [z : 1 : 0]$, which is an example of Proposition C.2. This in turn implies that $\mathbb{P}^1$ and $\{[z_1 : z_2 : z_3] \in \mathbb{P}^2 \mid z_3 = 0\}$ are conformally equivalent. Proposition C.2 gives us a great way to classify compact Riemann surfaces in a uniform manner and highlights the power of projective curves as Riemann surfaces.

The other compact Riemann surface we have covered in detail is the complex torus $\mathbb{C}/\Lambda$ for a lattice Λ. Before continuing with $\mathbb{C}/\Lambda$, we are going to take a sharp detour and discuss elliptic curves. Elliptic curves are algebraic curves of the form

$$\{(x_1, x_2) \in \mathbb{C}^2 \mid x_2^2 = \delta x_1^3 + \alpha x_1 + \beta\}.$$

with coefficients α, β and δ. In its current form, this algebraic curve is not particularly useful to us. However, if we set $x_1 = {}^{z_1}/_{z_3}$ and $x_2 = {}^{z_2}/_{z_3}$, then

$$\begin{aligned} x_2^2 = \delta x_1^3 + \alpha x_1 + \beta &\implies \left(\frac{z_2}{z_3}\right)^2 = \delta\left(\frac{z_1}{z_3}\right)^3 + \alpha\frac{z_1}{z_3} + \beta \\ &\implies z_2^2 z_3 = \delta z_1^3 + \alpha z_1 z_3^2 + \beta z_3^3 \end{aligned}$$

giving us a homogeneous polynomial in 3 variables, which can now be represented in the projective curve

$$\begin{aligned} E_{\delta,\alpha,\beta} &= \{[z_1 : z_2 : z_3] \in \mathbb{P}^2 \mid z_2^2 z_3 = \delta z_1^3 + \alpha z_1 z_3^2 + \beta z_3^3\} \\ &= \{[z_1 : z_2 : 1] \in \mathbb{P}^2 \mid z_2^2 = \delta z_1^3 + \alpha z_1 + \beta\} \cup \{[0 : 1 : 0] \in \mathbb{P}^2\}, \end{aligned}$$

allowing us to recover the elliptic curve format while also retaining the requirements for projective curves. We can set $z_3 = 1$ because the projective curve is invariant under non-zero scalar multiplication, and we can account for the isolated $z_3 = 0$ case by noticing the polynomial would become $\delta z_1^3 = 0$, implying $z_1 = 0$, giving us both components of the union. Now, let's see how this relates to the complex torus $\mathbb{C}/\Lambda$. Recall the Weierstrass $\wp$ function for the lattice Λ from Appendix A. The Laurent series of $\wp : \mathbb{C} \to \mathbb{C}$ is

$$\wp(z) = \frac{1}{z^2} + \sum_{j=1}^{\infty} a_{2j} z^{2j}, \text{ where } a_{2j} = (2j-1) \sum_{\lambda \in \Lambda\setminus\{0\}} \frac{1}{\lambda^{2j+2}}.$$

Let's consider the following Laurent expansions for $\wp(z)^3$ and $\wp'(z)^2$:

$$\wp(z)^3 = \frac{1}{z^6} + \sum_{j=-1}^{\infty} b_{2j} z^{2j},$$
$$\wp'(z)^2 = \frac{4}{z^6} + \sum_{j=-1}^{\infty} c_{2j} z^{2j},$$

with respective Laurent coefficients b_j and c_j. From observing these Laurent series, we can find coefficients α and β based on Λ such that

$$\wp'(z)^2 = 4\wp(z)^3 + \alpha\wp(z) + \beta.$$

If $z_1 = \wp(z)$ and $z_2 = \wp'(z)$, this would correspond to the polynomial in $E_{4,\alpha,\beta}$. Of course, these $\wp$ have the domain $\mathbb{C}$, not $\mathbb{C}/\Lambda$. In Appendix A, we were able to convert $\wp$ into a meromorphic function with the domain $\mathbb{C}/\Lambda$ since $\wp$ is doubly-periodic. Likewise, $\wp'$ is doubly-periodic, so we can also convert it to a meromorphic function on $\mathbb{C}/\Lambda$. From here, we can easily construct the biholomorphic mapping $f : \mathbb{C}/\Lambda \to E_{4,\alpha,\beta}$, where

$$f(z+\Lambda) = \begin{cases} [\wp(z+\Lambda) : \wp'(z+\Lambda) : 1] & \text{if } z \notin \Lambda \\ [0 : 1 : 0] & \text{otherwise,} \end{cases}$$

which shows conformal equivalence between $\mathbb{C}/\Lambda$ and a projective curve defined by an elliptic curve. We can map the lattice points to a single element $[0 : 1 : 0]$ since we identify each lattice point in $\mathbb{C}/\Lambda$, so the bijectivity of this mapping is not invalidated. This also implies that elliptic curves have genus 1 when considered as projective curves.

The coefficients α and β will be dependent on the lattice points in Λ, so the elliptic curve that represents each complex torus may differ. The big takeaway from this is that, while all complex tori are clearly topologically equivalent, they are not all conformally equivalent. For example, the torus defined by lattice points 1 and $\boldsymbol{i}$ will not be conformally equivalent to the torus defined by lattice points 1 and $1+\boldsymbol{i}$.

Proposition C.3

Let $\mathbb{C}/\Lambda_1$ and $\mathbb{C}/\Lambda_2$ be complex tori. $\mathbb{C}/\Lambda_1$ is conformally equivalent to $\mathbb{C}/\Lambda_2$ if and only if there exists $k \in \mathbb{C}^*$ such that $\Lambda_1 = k\Lambda_2$.

If $\Lambda = \{m\omega_1 + n\omega_2 \mid m, n \in \mathbb{Z}, \omega_1, \omega_2 \in \mathbb{C}^*\}$, then $k\Lambda = \{mk\omega_1 + nk\omega_2 \mid m, n \in \mathbb{Z}, \omega_1, \omega_2 \in \mathbb{C}^*\}$ for $k \in \mathbb{C}^*$. Essentially, this means that, if the angle between the lattice points (as vectors in $\mathbb{C}$) for each of the complex tori is preserved under non-zero scalar multiplication, only then will the complex tori be conformally equivalent. In our previous example, the angle between 1 and $\boldsymbol{i}$ would be $\pi/2$ but the angle between 1 and $1+\boldsymbol{i}$ would be $\pi/4$. This relates to the idea that holomorphic mappings need to preserve angles. This also immediately implies that not all compact Riemann surfaces of genus 1 are conformally equivalent.

C.2 Uniformisation

When discussing conformal equivalence, the Riemann mapping theorem from *MA3B8 Complex Analysis* is likely to come to mind. It states that every simply connected subset of $\mathbb{C}$ is conformally equivalent to the unit disc $\mathbb{D}$, except for $\mathbb{C}$ itself. Another way to view this is, under the conformal equivalence relation, there are two equivalence classes of simply connected subsets of $\mathbb{C}$. We can see that this extends to Riemann surfaces as we know that simply connected subsets of $\mathbb{C}$ are Riemann surfaces, but what other simply connected Riemann surfaces are there?

Lemma C.4

Let R be a compact Riemann surface. R is simply connected if and only if its genus is $g_R = 0$.

From our visualisation of the complex torus in Figure 3, if we had a closed path around the hole in the centre, we would not be able to shrink it to a constant path because the hole is in the way: since genera can be geometrically visualised as holes, this rules out all Riemann surfaces with positive genera as there will be closed paths on them which are not homotopic to constant paths. Of course, we are now left with genus 0 Riemann surfaces. From our visualisation of $\mathbb{C}_\infty$ in Figure 1, we can see that all closed paths are homotopic to constant paths: the Riemann sphere is simply connected. As we established earlier, $\mathbb{P}^1$ will also be simply connected as per the existence of the biholomorphic mapping between it and $\mathbb{C}$, which maintains topological properties. Any other compact Riemann surface with genus 0 will also be simply connected since they do not have any holes - we can easily find a homotopy from any closed path to a constant path.

So now our problem has been simplified to: which genus 0 Riemann surfaces are conformally equivalent to each other? The uniformisation theorem tells us about the equivalence classes of simply connected Riemann surfaces.

Theorem C.5 (Uniformisation Theorem) [7, p.185]

Let R be a simply connected Riemann surface. Then R is conformally equivalent to $\mathbb{D}$, $\mathbb{C}$ or $\mathbb{C}_\infty$.

The uniformisation theorem was initially discovered by Felix Klein and Henri Poincaré in the 1880s, and was later proved in 1907 by Poincaré and Paul Koebe [16, p.11]. It tells us that, amazingly, all genus 0 compact Riemann surfaces will be conformally equivalent to the Riemann sphere and, by transitivity, each other. This is because all genus 0 compact Riemann surfaces are simply connected, and since they are compact, and biholomorphic mappings are homeomorphisms, by the uniformisation theorem they can only be conformally equivalent to $\mathbb{C}_\infty$. This is not generally true, as we know that not all genus 1 Riemann surfaces are conformally equivalent to each other from Proposition C.3. The uniformisation theorem also highlights that there are only 3 sets of equivalence classes for simply connected Riemann surfaces, the two that the Riemann mapping theorem established, and the one induced by the Riemann sphere. In addition, there are no non-compact Riemann surfaces that are simply connected outside of $\mathbb{C}$ and its subsets.

We give names to each of these equivalence classes: a simply connected Riemann surface is hyperbolic if it is conformally equivalent to $\mathbb{D}$, parabolic if it is conformally equivalent to $\mathbb{C}$ and spherical if it is conformally equivalent to $\mathbb{C}_\infty$. Let's expand on one of the results we proved earlier while keeping in mind the uniformisation theorem.

Corollary C.6

Let R be a compact Riemann surface and $f : R \to \mathbb{C}$ be an injective meromorphic function. Then R is $\mathbb{C}_\infty$, up to conformal equivalence.

Proof. In Corollary 6.2, we already established that R has genus 0. From here we can use the uniformisation theorem to immediately see that, since R is simply connected (as all genus 0 compact Riemann surfaces are), it must be conformally equivalent to $\mathbb{C}_\infty$. □

We have achieved a powerful result utilising conformal equivalence and meromorphic functions, highlighting that R in this case will be a spherical Riemann surface. To finalise our discussion of conformal equivalence, let's explore some spherical Riemann surfaces. We have already seen three of these: $\mathbb{C}_\infty$, $\mathbb{P}^1$ and the projective curve $\{[z_1 : z_2 : z_3] \in \mathbb{P}^2 \mid z_3 = 0\}$. However, we can find out some more subtle spherical Riemann surfaces. Recall the formula for the genus of a Fermat curve F_d:

$$g_{F_d} = \frac{(d-1)(d-2)}{2},$$

which is 0 when the degree of the Fermat curve is either 1 or 2. Since we know genus 0 compact Riemann surfaces are simply connected and must be conformally equivalent to $\mathbb{C}_\infty$, we know that the Fermat curves F_1 and F_2 are also spherical Riemann surfaces. What would the biholomorphic mapping

$f : \mathbb{C}_\infty \to F_1$ be? We can rewrite the Fermat curve as

$$\begin{aligned} F_1 &= \{[z_1 : z_2 : z_3] \in \mathbb{P}^2 \mid z_1 + z_2 = z_3\} \\ &= \{[z_1 : z_2 : 1] \in \mathbb{P}^2 \mid z_1 + z_2 = 1\} \cup \{[1 : -1 : 0] \in \mathbb{P}^2\}, \end{aligned}$$

similar to our approach when we were working with elliptic curves. Using this format, we can see that

$$f(z) = \begin{cases} [1 : -1 : 0] & \text{if } z = \infty \\ [z : 1 - z : 1] & \text{otherwise,} \end{cases}$$

is a biholomorphic mapping which fulfills the polynomial in the Fermat curve of degree 1.

The uniformisation theorem is incredibly powerful in the discussion of conformal equivalence and simple connectedness, allowing us to make some immediate judgements on the relation of differing Riemann surfaces.

D Additional Proofs

Here I have provided additional proofs for some of the propositions and theorems mentioned earlier. The areas covered here are:

- continuity of holomorphic mappings,
- the open mapping theorem,
- k-sheeted covering maps.

D.1 Continuity Proof

Proposition D.1

Let R and S be Riemann surfaces and $f : R \to S$ be a holomorphic mapping. Then f is continuous.

Proof. Let $\{(\phi_\alpha : U_\alpha \to V_\alpha, U_\alpha)\}_\alpha$ be a holomorphic atlas for R and $\{(\psi_\beta : W_\beta \to X_\beta, W_\beta)\}_\beta$ be a holomorphic atlas for S. Firstly, let's prove continuity at a point $w \in R$. Using the pair of charts (U_α, ϕ_α) and (W_β, ψ_β) containing w and $f(w)$ respectively, consider

$$\psi_\beta \circ f \circ \phi_\alpha^{-1} : V_\alpha \cap \phi_\alpha(f^{-1}(W_\beta)) \to X_\beta.$$

This is a holomorphic function on a subset of $\mathbb{C}$, meaning it is complex differentiable and so is continuous on V_α and therefore continuous on $V_\alpha \cap \phi_\alpha(f^{-1}(W_\beta))$. Since ψ_β and ϕ_α are homeomorphisms, f is continuous on U_α meaning it is continuous at w. Since this applies to all points in R, f is continuous. □

D.2 Open Mapping Theorem Proof

Theorem D.2 (Open Mapping Theorem) [8, p.10]

Let R be a Riemann surface and $f : R \to S$ be a non-constant holomorphic mapping. Then f is an open map.

Proof. Let $\{(\phi_\alpha : U_\alpha \to V_\alpha, U_\alpha)\}_\alpha$ be a holomorphic atlas for R and $\{(\psi_\beta : W_\beta \to X_\beta, W_\beta)\}_\beta$ be a holomorphic atlas for S. Since f is a non-constant holomorphic mapping, by the standard open mapping theorem (from *MA3B8 Complex Analysis*), $\psi_\beta \circ f \circ \phi_\alpha^{-1}$ is an open map, as it is a holomorphic function. As ψ_β and ϕ_α are homeomorphisms, $f|_{U_\alpha}$ must also be an open map. As the arbitrary union of open sets is open, and this applies to every chart, f must be an open map on the union of charts within the holomorphic atlas for R and therefore we know f is an open map. □

D.3 Covering Map Proof

Proposition D.3 [10, p.13]

Let R and S be compact Riemann surfaces and $f : R \to S$ be a non-constant holomorphic mapping. Let $S' = S \setminus \mathrm{Br}(f)$ and $R' = R \setminus \mathrm{Va}(f)$, so $f(R') = S'$. Then $f|_{R'} : R' \to S'$ is a $\deg(f)$-sheeted covering map.

Proof. Firstly, let's show that $f|_{R'}$ is a compact map. Any compact set $K \subset S'$ is a compact set in S. We know that f is a compact map by Proposition 3.11, so $f^{-1}(K)$ is also compact. Since $f^{-1}(K) \subset R'$ as $K \subset f(R')$, $f|_{R'}^{-1}(K) = f^{-1}(K)$, so $f|_{R'}^{-1}(K)$ is compact (note that this is NOT $f^{-1}|_{R'}$). This means $f|_{R'}$ is a compact map.

Since we have removed all the valency points of f to obtain $f|_{R'}$, all $w \in R'$ will have the local representation z under $f|_{R'}$. By Corollary 3.12, for $s \in S'$, as $f|_{R'}$ is a compact map, $|f|_{R'}^{-1}(s)|$ is finite.

Let $V \subset S'$ be a very small open set with $s \in V$. If

$$|f|_{R'}^{-1}(s)| = |f^{-1}(s)| = \{w_1, \dots, w_n\},$$

take $\{U_1, \dots, U_n\}$ to be a set of non-intersecting open sets of R' (so they are all disjoint), such that each $w_\alpha \in U_\alpha$ and $f|_{R'}(U_\alpha) = V$ (we choose V to be very small to guarantee no intersections between the U_α). As the local representation of each point is z, a local inverse exists everywhere, so $f|_{U_\alpha}$ is a homeomorphism for all U_α. In addition,

$$f|_{R'}^{-1}(V) = \bigsqcup_{\alpha=1}^{n} U_\alpha,$$

so $f|_{R'}$ is a covering map. To confirm that $f|_{R'}$ is a $\deg(f)$-sheeted covering map, we can write $|f|_{R'}^{-1}(s)|$ in an unorthodox way:

$$|f|_{R'}^{-1}(s)| = |f^{-1}(s)| = \sum_{\substack{w \in R \\ f(w)=s}} 1.$$

We know that $v_f(w_\alpha) = 1$ for all $w_\alpha \in f^{-1}(s)$ so we can write

$$|f|_{R'}^{-1}(s)| = |f^{-1}(s)| = \sum_{\substack{w \in R \\ f(w)=s}} 1 = \sum_{\substack{w \in R \\ f(w)=s}} v_f(w) = \deg(f),$$

and, by the constancy of $\deg(f)$, this will apply for any $s \in S'$, implying that $f|_{R'}$ a $\deg(f)$-sheeted covering map. $\square$

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