Detecting Fibred Links and Computing Monodromy

Outline

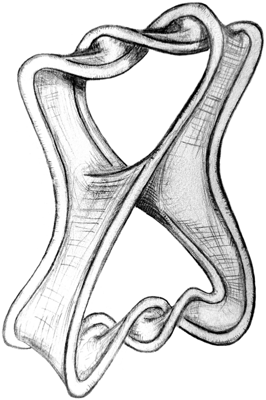
Seifert Surfaces

Seifert surface for an oriented link $L \subset S^3$:

- Connected compact oriented surface R;
- $\partial R = L$.

The orientation on R defines positive and negative sides of $R \times I$ which meet along a thickened copy of the link. *(Sutured product)*

Examples of Seifert Surfaces



Seifert surface for the figure-eight.

Thickened Seifert surface for the right-handed trefoil.

Seifert Algorithm

A Seifert surface always exists. *(Seifert algorithm)*

- Take any diagram of the link;
- Find *Seifert circles*;
- Get a collection of discs from the Seifert circles;
- Attach a band between discs for each crossing.

Genus

The genus of a knot K is the least genus g for which a genus g Seifert surface for K exists.

Example

There are genus one Seifert surfaces for the trefoil (Rolfsen 3_1) and figure-eight (Rolfsen 4_1) knots. Since these knots are not trivial, they cannot be genus zero. So the trefoil and figure-eight are genus one.

Fibration

A link is *fibred* if the complement in S^3 *fibres over the circle.*

Formally, a map $f : E \to B$ is a *fibration* with fibre F if each point of B has a neighbourhood U and a trivialisation $f^{-1}(U) \to U \times F$ such that

$$\begin{array}{ccc} f^{-1}(U) & \longrightarrow & U \times F \\ & \searrow & \swarrow_{\mathrm{proj}_1} \\ & U & \end{array}$$

commutes.

Fibred Links

A link $L \subset S^3$ is *fibred* if there exists a fibration

$$f\colon S^3 \setminus L \to S^1$$

such that each component of L has regular neighbourhood $S^1 \times D^2$ where the restriction of f to $S^1 \times (D^2 \setminus 0)$ is given by

$$(x, y) \mapsto y/|y|$$

(well-behaved near L).

It follows that $f^{-1}(x) \cup L$ is a Seifert surface for L for all $x \in S^1$. *(Closure of each fibre is a Seifert surface.)*

Monodromy

A fibration over the circle is determined by a homeomorphism of the fibre.

- Take a fibration $E \to S^1$ with fibre F and pull back along $[0,1] \to S^1$ to get a fibration over $[0,1]$.
- Since $[0,1]$ is contractible, any fibration over $[0,1]$ is trivial.
- So E pulls back to $F \times [0,1]$.
- Hence there is a homeomorphism $h \colon F \to F$ gluing $F \times \{0\}$ to $F \times \{1\}$ so that $E \cong F \times [0,1]/\sim$.

The fibration is said to have *monodromy* h. *(Not unique.)*

Checking if a Seifert Surface is a Fibre

How do we know if a given Seifert surface is a fibre for some fibration?

To check if a Seifert surface R for L is a fibre:

- Thicken the Seifert surface into $R \times I$;
- Take the complement of $R \times I$ in S^3;
- If the complement is also a product, then the R is a fibre of L.

In *Detecting fibred links in S^3* (1986), David Gabai presents a simple method for detecting that R is a fibre using decompositions along *product discs* in the complement.

Sutured Manifolds

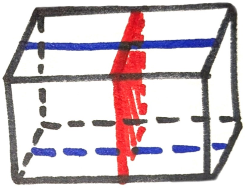
A *sutured manifold* is a pair (M, γ) where:

- M is a compact oriented 3-manifold;
- $\gamma \subset \partial M$ is a collection of disjoint simple closed curves;
- The curves can be thickened and are called sutures;
- The sutures divide ∂M into surfaces $R_\pm$ with shared boundary γ;
- The surfaces $R_\pm$ are oriented oppositely and γ has the induced orientation.

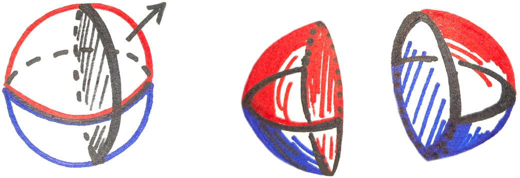
Think of the positive surface R_+ as having an outward normal vector and the negative surface R_- as having an inward normal vector.

Pictures of Sutured Manifolds

Sutured ball $D^2 \times I$.



Product disc.



Decomposing $D^2 \times I$ along a product disc.

Product Sutured Manifold

A sutured manifold (M, γ) is a *product* if:

- $M = R \times I$ and the sutures thicken to $\partial R \times I$;
- R is a compact oriented surface with no closed components.

Product Disc

A oriented disc D in (M, γ) is a *product disc* if:

- $D \subset M$ is proper;
- $|D \cap \gamma| = 2$.

The existence of a product disc in (M, γ) tells us that the manifold looks like a product in a neighbourhood of the disc.

Product Decomposition

Let D be a product disc in (M, γ). The *product decomposition* along D is (M', γ') where:

- M' is obtained from M by cutting along D;
- Any point where the normal orientations disagree is regarded as lying in a suture;
- The new sutures are γ'.

Product Decompositions Preserve Products

Lemma

Let D be a product disc in (M, γ) and let (M', γ') be the product decomposition along D. Then (M', γ') is a product if and only if (M, γ) is a product. (Gabai 1986)

Proof.

(if)

- Suppose (M, γ) is $R \times I$.
- Isotope D to be of the form $\alpha \times I$ where α is a proper arc in R.
- Cutting along $D = \alpha \times I$ gives $M' = (R \setminus \mathring{N}(\alpha)) \times I$.

(only if)

- Suppose (M', γ') is $R' \times I$.
- M is recovered from M' by gluing back in a thickening of D.
- (M, γ) is $R \times I$ where R is constructed from R' by attaching a band.

□

Checking if a Sutured Manifold is a Product

Theorem

A sutured manifold (M, γ) is a product if and only if there is a sequence of product decompositions that terminates in $E \times I$ where E is a union of discs. *(Gabai 1986)*

Proof.

(if)

- Immediate from the previous lemma.

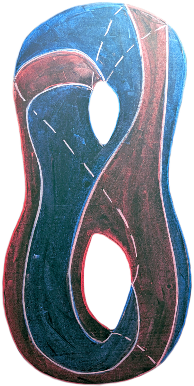
(only if)

- Suppose (M, γ) is $R \times I$;
- Choose a family of (pairwise disjoint) proper arcs that cut R into a union of disc;
- A sequence of product decompositions along discs $D_i = \alpha_i \times I$ results in a union of copies of $D^2 \times I$.

□

Examples of Fibred Knots

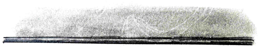
The trefoil is fibred.



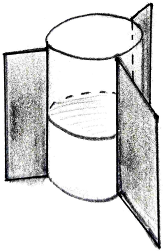
The figure-eight is fibred.

Constructing a Fibration of the Trefoil

Begin with a fibration of the unknot.



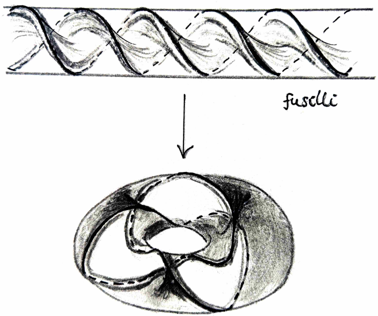
A single page that rotates around the unknot.



Three pages and a meridional disc.

The trefoil has three-fold symmetry, so we want three pages.

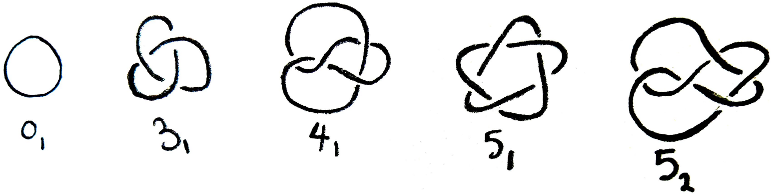
Constructing a Fibration of the Trefoil



Rolfsen 10.I page 327

Similarly, every *torus knot* (or, more generally, *cable knot*) is fibred with fibration looking like fusilli pasta (a corkscrew).

Example of a Non-fibred Knot



Rolfsen knot table

The 5_2 knot is the first non-fibred knot in the Rolfsen knot table. *(The commutator subgroup of the fundmamental group is not finitely generated.)*

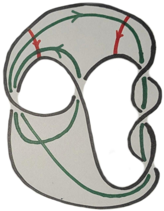
Computing the Monodromy

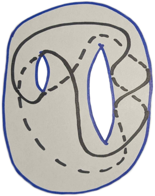
To see *how* the complement of a fibred link is fibred, we can look at how curves flow from one side of the fibre surface to the other under action of going around S^1.

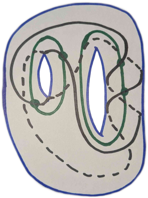
As a fibre surface R turns around the link in S^3, any arc α with its endpoints fixed on ∂R is dragged from one side of R to the other. It takes a new position when it returns to R and this new position is exactly the image of the arc under the monodromy.

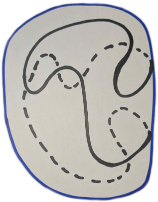
We can construct a piece of the flow by repeatedly sending an arc around, letting it drag a surface behind.

Pictures for the Figure-eight









Monodromy of the Trefoil and Figure-eight

The trefoil and figure-eight knots have fibrations with genus one fibres. So their monodromies can be described by elements of $\mathrm{SL}_2(\mathbb{Z})$. *(Mapping class group.)*

Each element A of $\mathrm{SL}_2(\mathbb{Z})$ is either:

- periodic *(where* $|\operatorname{tr}(A)| < 2$*)* or
- *reducible (where* $|\operatorname{tr}(A)| = 2$*)* or
- *Anosov (where* $|\operatorname{tr}(A)| > 2$*)*.

The trefoil has periodic monodromy whereas the figure-eight has Anosov monodromy.