Pasta!

ies

Detecting Fibred Links and Computing Monodrom~~y~~

singular/plural agreement?

by

[redacted]

MA4K9 Dissertation

Submitted to The University of Warwick

Mathematics Institute

April, 2025

I prefer “three-manifolds” - because I prefer writing out small numbers.
It is a personal choice.... but see Bjorn Poonen’s rules for writing here:
https://math.mit.edu/~poonen/papers/writing.pdf
(which are excellent in any case).

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1 Introduction

Thurston's geometric point of view on 3-manifolds was built on intuition from a repertoire of constructions and examples [Thu94, page 14]. One example that plays an important role in his 1980 notes [Thu80] is the complement of the figure-eight knot (a genus one fibred knot with Anosov monodromy, see Figure 1). Inspired by the final topological picture story of Francis' *A Topological Picturebook* [Fra87], the purpose of this project is to develop the tools needed to visualise link complements that fibre over the circle, with a focus in Section 5 on the case of genus one fibres (where an attempt is made at "seeing" the fibres and the flow).

Very good

good - this is the "thesis" sentence of the essay.

A formal development of knot theory requires familiarity with the technicalities of either piecewise linear or differential topology. Our approach will be to assume some basic knowledge of piecewise linear topology as discussed in Chapters 2 and 4 "of" ~~in~~ [Rol76].

1.1 Summary of Results

There are two main parts to "our" ~~the~~ story: detecting if a link is *fibred*; and understanding "its" ~~the~~ *monodromy*. The fibres of a fibred link are oriented surfaces that span the link they turn around under the flow. The monodromy is the homeomorphism given by flowing a point through the link complement until it first returns to the same fibre it began on.

run-on sent.

The key idea in Section 3 is the definition of the infinite cyclic cover X_∞ (see Definition 3.4). This is a cover of the exterior of a link that has great value as a geometric tool. This tool is utilised in the proof of Theorem 3.6 which states that any two Seifert surfaces for the same link can be modified by surgery to be isotopic.

The first part of the story begins in Section 4 where product decomposition of sutured manifolds gives a convienient method for checking if a link is fibred. The second part begins in 5 where the homeomorphisms of surfaces are studied with a focus on the genus one case. but punctured, right?

We begin in Section 2 with some basic definitions that will be used throughout.

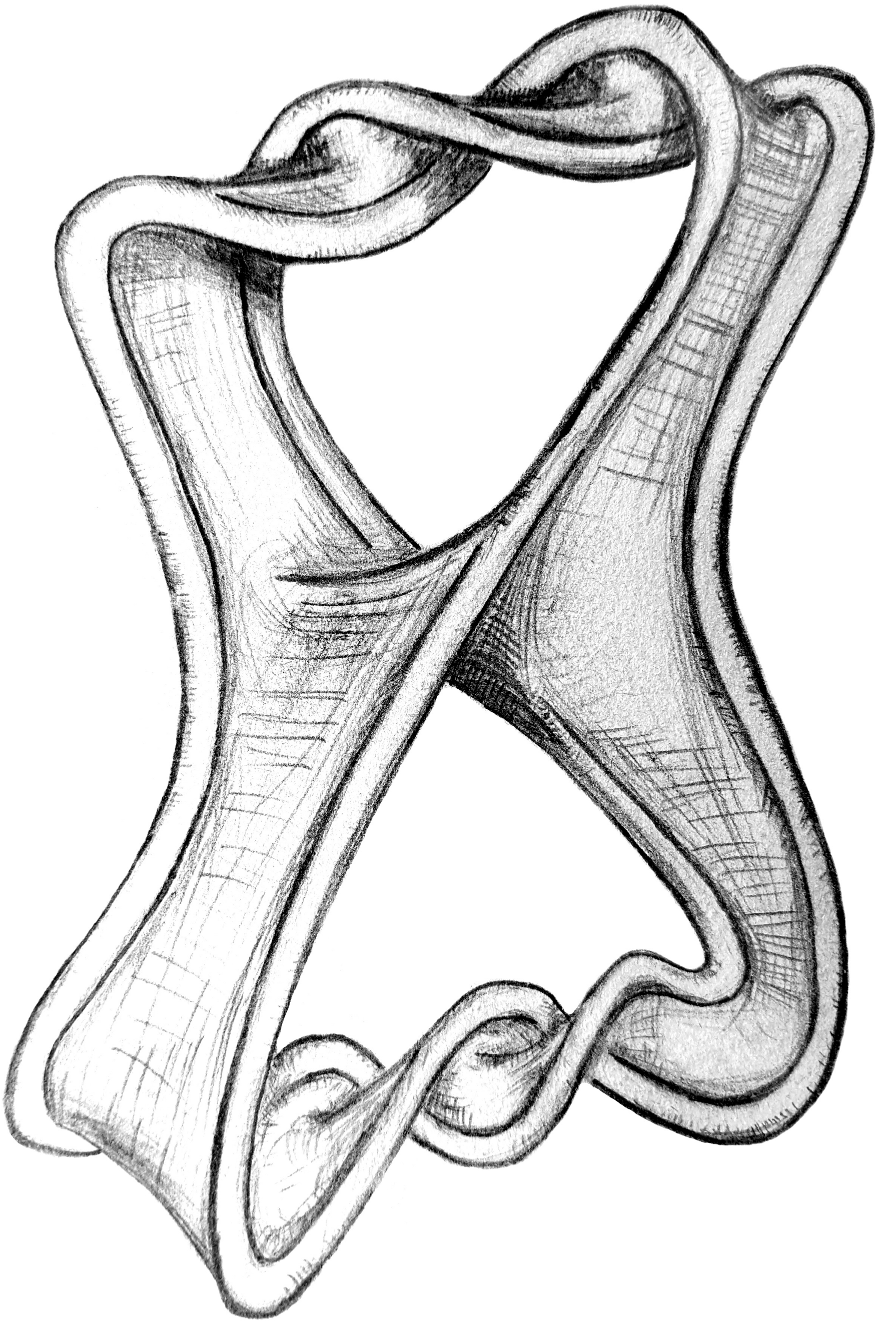


Figure 1: A minimal spanning surface for the figure-eight knot. [Fra87, page 37]

"After" or "following" as it is not a literal copy.

Question: Please sketch the proof (of 2.1) with some pictures.

Question: Please explain why Figures 2, 3 cannot be smooth.

“being wild: an example is shown in…”

2 Knots and 3-Manifolds

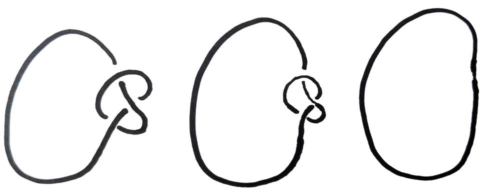
A *knot* is an embedding of S^1 in S^3 that can be *thickened* to an embedding of $S^1 \times D^2$. Being able to extend the embedding in this way prevents the knot from ~~looking like the embedding in~~ Figure 2. We also need to be careful about what it means for knots to be equivalent. As shown in Figure 3, any knot can be continuously deformed to the *unknot*.

A *link* is an embedding of $S^1 \times \{1, \ldots, n\}$ that can be thickened.

Figure 2: Embedding that is not sufficiently smooth.

When a homotopy is through embeddings or homeomorphisms it is called an *isotopy*. There is stronger notion called *ambient isotopy* which requires that the isotopy is carried out by a family of homeomorphisms of the ambient space.

Formally, embeddings $f, g \colon X \to Y$ are *ambient isotopic* if there is a map $G \colon Y \times I \to Y$ with G_t a homeomorphism for all $t \in I$, $G_0 = \mathrm{id}_Y$ and $G_1 \circ f = g$.



could be clearer about degrees of smoothness.

Figure 3: Isotopy that is not sufficiently smooth.

The isotopy extension theorem states that smooth isotopies extend to ambient isotopies.

Theorem 2.1. *Let X be a compact submanifold of a smooth manifold Y. Any smooth isotopy $F \colon X \times I \to Y$ of embeddings extends to an ambient isotopy $G \colon Y \times I \to Y$ satisfying $G_t \circ F_0 = F_t$. [Wal16, section 2.4].*

what is M? X?

Sketch proof. [Juh23, theorem 1.4.3] The idea is to extend a flow in $\boxed{M} \times I$ that comes from the isotopy. An isotopy $F \colon \boxed{S} \times I \to M$ flows $F_0(S) \times \{0\}$ through the product $M \times I$ to $F_1(S) \times \{1\}$. This flow extends to all of $M \times I$ by defining it to be parallel outside the tube swept out by $F_0(S) \times \{0\}$ under the flow. An ambient isotopy $G \colon M \times I \to M$ is defined by following the flow from $(x, 0)$ until we reach $M \times \{t\}$. □

S is Y?

Question: Give an example of.a Mobius band properly embebdded in a solid torus.
Does this have a bicollar?

a bit vague

For us, a *manifold* will be a metric space M that is locally modelled on $\mathbb{R}^{n-1} \times [0, \infty)$. The metric on M is frequently used (implicitly) to construct well-behaved neighbourhoods of submanifolds. Also, submanifolds $N \subset M$ will often be *proper*, meaning that the intersection of N with ∂M is exactly ∂N, N meets ∂M transversely, and every compact set in M intersects N in a compact set. Better to take M and N to be compact at the outset.

We will also need use of *triangulations*. A *triangulation* of a manifold M is a homeomorphism from a simplicial complex to M. The existence of triangulations in low dimensions is a theorem of Moise, Bing and Radó. Every compact manifold of dimension $n \leq 3$ is homeomorphic to a finite simplicial complex [Moi77]. Pinpoint ref is much better.

2.1 Neighbourhoods

"the idea of"

A thickening of an embedded submanifold $S \subset M$ is formalised by a *regular neighbourhood*.

Definition 2.2. *A regular neighbourhood $N(S)$ of S is an embedding $t\colon S \times D^k \to M$ where k is the codimension of S and $t(x, 0) = x$ for all $x \in S$.*

too much italics

orientation issues...

The special case of a regular neighbourhood for a codimension one submanifold is called a bicollar. Such a neighbourhood $t\colon S \times [-1, 1] \to M$ gives positive and negative sides to S in M. If a codimension one submanifold S is contained in the boundary of M then it cannot admit a bicollar, but it may admit a collar neighbourhood $t\colon S \times [0, 1] \to M$.

in fact always does

It will always be assumed that submanifolds can be thickened in the sense that they admit a regular (or collar) neighbourhood.

2.2 Triangulation and Orientation

An orientation on a triangulated 3-manifold M is a choice of orientation on each 3-simplex such that any two 3-simplices sharing a face have consistent orientations. a bit vague

Let S be a proper oriented surface in an oriented triangulated 3-manifold M such that S is a subcomplex. Every 2-simplex in M either lies on the boundary ∂M, or is in the interior and therefore a face of exactly two 3-simplices. A 2-simplex in the interior has induced orientations from 3-simplices on both sides and these orientations are inconsistent. So S is *two-sided* with the positive and negative sides of S defined by which side of each 2-simplex of S induces the correct orientation. The normal orientation of S is given by the normal vectors pointing from the negative to the positive side. [Sch90, p. 1.3]

Definition 2.3. *The fundamental class $[S] \in H_k(M, \partial M)$ of a proper oriented submanifold $S \subset M$ that is also a subcomplex is the sum over the top-dimensional simplices of S.*

All manifolds will be triangulated and oriented but a triangulation will only be used explicitly in Theorem 3.6. It will often be implicitly assumed that homeomorphisms are piecewise linear.

very nice!

Figure 4: Model of a Seifert surface for the right-handed trefoil knot.

Question: Give a Seifert surface for the third twist knot (5_2).

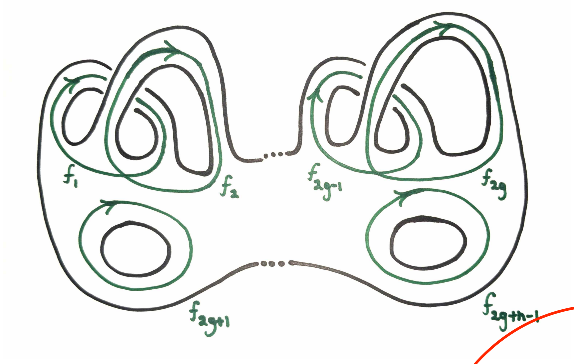
Question: Explain the phrase “is a finite set of points” used below.

3 Seifert Surfaces

“in the three-sphere S^3”

Definition 3.1. *A Seifert surface for an oriented link L is a connected compact oriented surface F in S^3 with no closed components and with boundary $\partial F = L$.*

By the classfication of surfaces, the homeomorphism type of a Seifert surface is classified by its genus and number of boundary components [Rol76, 5.A1]. A connected surface F of genus g and with n boundary components can be constructed from $2g+n-1$ bands, as in Figure 5. A basis for the first homology group $H_1(F) = \mathbb{Z}^{2g+n-1}$ is given by the oriented simple closed curves f_i depicted in Figure 5.



“modelled on”
not a literal copy of the figure…

Figure 5: Generators for the homology of F. [Lic97, figure 6.1]

The obvious Seifert surface for the unknot is the disc it bounds in the plane. The disc has the minimal possible genus so we say the unknot is *genus zero*. In general, the *genus* of a link is defined to be the least genus of any Seifert surface for the link. Examples of genus one knots include the left and right trefoil and the figure-eight knot. Better with a citation.

3.1 Intersection form

Recall that Poincaré–Lefschetz duality gives an isomorphism

$$D\colon H^k(M,A) \to H_{n-k}(M,B)$$

for a compact oriented n-manifold whose boundary ∂M is a union of two compact submanifolds A and B [Hat02, theorem 3.43]. Denoting the Poincaré dual of a homology class α by α^*, the *intersection form* [Hut11]

This looks wrong to me.

$$H_{n-i}(M,\partial M) \times H_{n-j}(M,\partial M) \to H_{n-i-j}(M,\boxed{\partial M})$$

is defined by $\alpha \cdot \beta = (\alpha^* \smile \beta^*)^*$. When $i+j=n$ the intersection $\alpha \cdot \beta$ is a finite set of points. The total signed number of intersection points is the *intersection number* of α and β. We will see that intersection numbers are related to *linking numbers*.

no…
homology classes are not points…

If K is a knot in S^3 then Alexander duality [~~See~~ Hat02, corollary 3.45] implies that the *knot complement* has $H^1(S^3 \setminus K) \cong \mathbb{Z}$. The *knot exterior* $X = S^3 \setminus \mathring{N}(K)$ has the same (co)homology because it is homotopy equivalent.

Theorem 3.2. *Any two Seifert surfaces for an oriented knot are homologous in the knot exterior.*

Proof. Let K be a knot with Seifert surface F. Suppose the orientaton on K is induced by the orientation on the surface F. By Poincaré–Lefschetz duality the knot exterior X has $H_2(X, \partial X) \cong H^1(X) \cong \mathbb{Z}$. When intersecting a closed curve and a properly embedded surface, the intersection form is

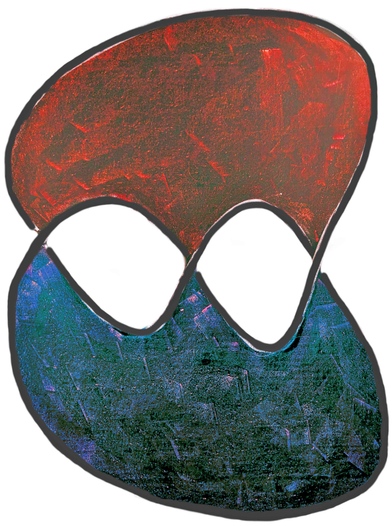
This is zero, so there is some mistake here.

$$H_1(X) \times H_2(X, \partial X) \to \boxed{H_0(X, \partial X)}$$

and all these homology groups are $\mathbb{Z}$. If α is a meridian loop for K, the intersection number with F is ± 1. Hence we must have that $[F]$ is a generator for $H_2(X, \partial X)$ by bilinearity of the intersection form. Which generator it is only depends on the choice of orientation. $\square$

Corollary 3.3. *The intersection number of a closed curve α in the knot exterior X and a Seifert surface F is independent of the choice of Seifert surface.* \qed

good This intersection number is exactly the *linking number* [Rol76, p. 5.D] of the curve and the knot. The intersection number of a closed curve and a Seifert surface for a link is the sum of the linking numbers with each component.



good

Figure 6: Positive and negative sides of a Seifert surface.

Question: Explain why the cover is normal.

3.2 Infinite cyclic cover

Let L be an oriented link in S^3 and F a Seifert surface for the link. Removing an open regular neighbourhood of L from S^3 gives the exterior X, which is a compact manifold with torus boundary components. The intersection $F \cap X$ is a copy of F with a collar neighbourhood of the boundary ∂F removed. Let U be the open neighbourhood found by intersecting X with the interior of a bicollar $F \times [-1, 1]$ for $F \subset S^3$.

Too much italics

Definition 3.4. *[Rol76, p. 5.C] Cut X along F to get the compact manifold $Y = X \setminus U$ whose boundary consists of two copies of F, denoted F^- and F^+, that meet along a thickened copy of the link. Note that X can be recovered* up to homeo *from Y by gluing F^- to F^+ via the homeomorphism ϕ found by pushing F^- across the bicollar $F \times [-1, 1]$ to F^+. The infinite cyclic cover X_∞ is constructed by gluing together countably infinite copies of Y, gluing each F^- to the next F^+ via ϕ. So X_∞ is a cover of X with deck group generated by the homeomorphism $t \colon X_\infty \to X_\infty$ that translates each copy of Y once in the positive direction.*

Since the covering $p \colon X_\infty \to X$ is normal (why?), the deck group $\mathbb{Z}$ is isomorphic to $\pi_1(X, x)/H$ where H is the image of $\pi_1(X_\infty, \tilde{x})$ under the induced map p_* and consists of the loops in X based at x whose lifts to X_∞ starting at $\tilde{x}$ are loops. The Galois theory of covers states that the coverings of X are classified by the subgroup H [Hat02, section 1.3].

Lemma 3.5. *The covering $p \colon X_\infty \to X$ does not depend on the choice of Seifert surface F used in the construction. [Lic97, theorem 7.9]*

Proof. Let L be a link and F a Seifert surface for L. Any loop $\alpha \colon I \to X$ at a point in X lifts to path in X_∞ whose endpoints are in the fibre over that point. Every time α crosses F, its lift passes from one copy of Y to another. So α lifts to a loop in X_∞ exactly when the intersection number of α and F is zero. That is, exactly when the sum of the linking numbers with the components of L is zero. This statement is independent of the choice of Seifert surface F. So the group of the cover does not depend on F and the result follows from the Galois theory of covers. $\square$

The next theorem improves on Theorem 3.2. If two surfaces in S^3 are homologous then they are related by operations that replace $D^2 \times \partial I$ with $\partial D^2 \times I$ and vice versa. When everything is embedded, these are the operations of *surgery* and *compression*.

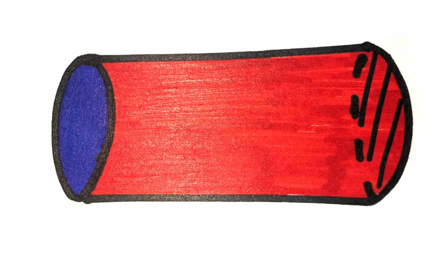


Figure 7: Boundary of $D^2 \times I$.

If two Seifert surfaces F and F' for a link L have intersection L then $F \cup F'$ is a closed orientable surface in S^3 and therefore must bound a handlebody M. The proof of the next theorem uses a procedure that fills M with intertwined handles such that the new surfaces constructed are isotopic. No

Theorem 3.6. *Any two Seifert surfaces F and F' for an oriented link L in S^3 are related by a sequence of isotopies, surgeries, and compressions. [Lic97, theorem 8.2]*

Proof. It may be assumed, by a small isotopy, that F and F' intersect transversely in finitely many simple closed curves, including their common boundary L. Suppose that F and F' intersect away from L. It will be shown that the number of components of the intersection of the two Seifert surfaces can be reduced until $F \cap F' = L$.

Let X be the exterior of the link L and construct the infinite cyclic cover X_∞ by cutting along F. There are infinitely many copies of F' (with a collar neighbourhood of the boundary removed) in X_∞ found as lifts of F' (again, without its boundary). Fix lifts of F and F' in X_∞ which are denoted the same for simplicity. Let n be the maximal integer such that $t^n F \cap F'$ is non-empty. The surface F' separates X_∞ into two components, one on the negative side and the other on the positive side. We are interested in what happens in the space between $t^n F$ and F'.

So let Y_n be the copy of Y between $t^n F$ and $t^{n+1} F$ and consider the intersection of Y_n with the negative side of F' in X_∞. The closure M of any component of this intersection is a compact manifold that lies between $t^n F$ and F' and whose boundary is contained in $t^n F \cup F' \cup \partial X_\infty$. Since M lies inside a copy of Y it projects to a copy of itself (again, denoted the same) in X. Moreover, this copy of M in X is just the closure of some component of $X \setminus (F \cup F')$.

We are going to use the manifold M to decide how to change F and F'. The important properties of M are that it is the closure of some component of $X \setminus (F \cup F')$ and whenever M meets either F or F' it does so from the same side. This second property makes sure that F and F' stay orientable under the surgeries that follow.

Write ∂M as the union $A \cup A'$ of the compact surfaces $A = \partial M \cap F$ and $A' = \partial M \cap F'$ along their shared boundary $A \cap A'$. Pick a triangulation of S^3 with A, A' and M as

Question: Explain why the "grand stabilisation" can be obtained by (the opposite of) surgeries (and isotopies).

subcomplexes. The part of $F \cap F'$ that we are trying to remove is $A \cap A'$. It will be important that $A \cap A'$ is contained in the 1-skeleton of M. Let B be a collar neighbourhood of $A \subset M$ together with a regular neighbourhood (in M) of the 1-skeleton of M and let B' be the closure of $M \setminus B$.

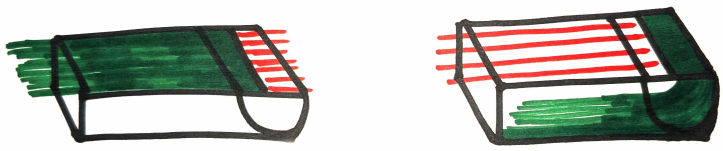


Figure 8: The local picture near $A \cap A'$.

Change F by removing A and replacing it with the closure of $\partial B \setminus A$. Change F' by removing $B' \cap F'$ and replacing it with the closure of $\partial B' \setminus (B' \cap F')$. These changes can be achieved by isotopies and surgeries.

The new intersection of F and F' contains $\partial B \setminus A$. The closure of $\partial B \setminus A$ contains $\partial A = A \cap A'$. A small isotopy removes the closure of $\partial B \setminus A$ from the intersection, thereby reducing the number of components. In this way we can reduce the number of components in the intersection until $F \cap F' = L$.

Finally, we note that the procedure described produces surfaces that are isotopic inside M. So when F and F' only intersect along the link L we can apply the procedure similarly to get isotopic surfaces. $\square$

4 Fibred Links

4.1 Sutured Manifolds

In Definition 3.4 we saw that the exterior of a Seifert surface is a compact 3-manifold whose boundary can be decorated with a thickening of the link. Decorations such as these will be called *sutures*. The boundary of the manifold is separated into positive and negative sides by the sutures.

Definition 4.1. *A sutured manifold is a pair (M, γ) where:*

1. *M is a compact oriented 3-manifold;*
2. *$\gamma \subset \partial M$ is a (possibly empty) collection of disjoint simple closed curves;*
3. *The curves can be thickened and are called sutures;*
4. *The sutures divide ∂M into two surfaces $R_\pm$ with shared boundary γ;*
5. *The surfaces $R_\pm$ are oriented oppositely and γ has the induced orientation.*

The definition of a sutured manifold is due to Gabai and first appeared in his thesis (1980) [Gab84] which was supervised by Thurston.

We can think of the positive surface R_+ as having an outward normal orientation and the negative surface R_- as having an inward normal orientation. The sutures keep track of where the normal orientations on ∂M flip.

“is”

If R be a compact oriented surface with no closed components (such as a Seifert surface) then $R \times I$ is a sutured manifold with collection of sutures given by $\partial R \times I$. A sutured manifold of this form is called a *product*. The choice of which side is positive and which is negative is decided by the convention that the normal orientation on R points from the negative to the positive.

Example 4.2. *The disc is a Seifert surface for the unknot. The product $D^2 \times I$ is a ball with a single suture (the unknot). The complement in S^3 is another ball.*

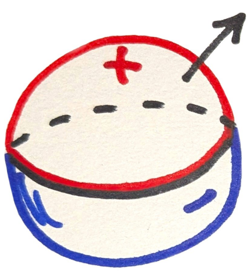
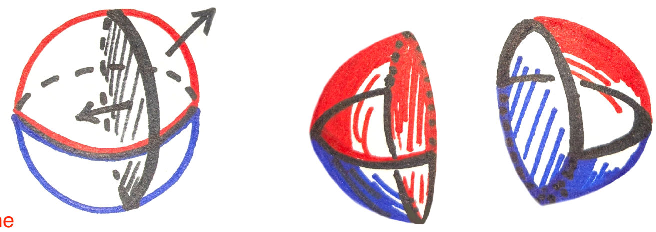


Figure 9: Sutured ball.

In the ball $D^2 \times I$ there are proper discs that intersect the suture exactly twice. In Figure 10, close attention is paid to the normal orientations during the process of cutting along one of these *oriented* discs to produce two copies of $D^2 \times I$.



This looks like a mistake - the normal on the black product disk should point the other way.

Figure 10: Decomposition along a disc.

It is important that the thickened disc has positive and negatives sides (in fact it is another copy of the sutured ball). After decomposing along the disc, any point where the normal orientations disagree is regarded as lying in a suture [Sch90, p. 6.1]. In general, a *product disc* in (M, γ) is a proper disc $D \subset M$ such that $|D \cap \gamma| = 2$ and decomposing along a product disc, with the new sutures defined in this way, is a *product decomposition*.

The effect of a product decomposition on the subsurfaces $R_\pm$ can be described by a simple geometric operation between each subsurface (pushed slightly into M relative to its boundary) and the product disc. This operation is the *double curve sum* and is depicted in Figure 11. For oriented surfaces S and T in general position it is defined by removing a neighbourhood of the intersection $S \cap T$ and connecting the resulting surfaces in a way that is consistent with the orientations.

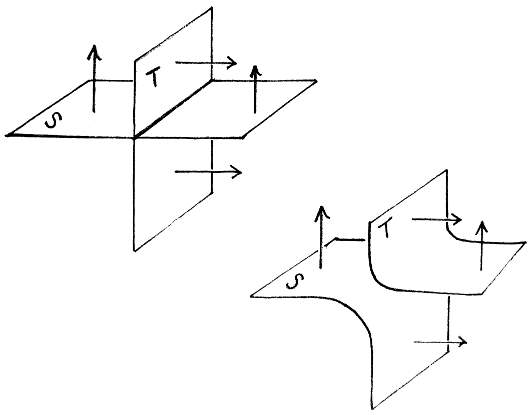


Figure 11: Double curve sum. [Sch90, p. 5.3]

A sutured manifold can be decomposed along any proper oriented surface. However, for our purposes product discs are enough.

Lemma 4.3. *Let D be a product disc in (M, γ) and let (M', γ') be the product decomposition along D. Then (M', γ') is a product if and only if (M, γ) is a product. [Gab86, lemma 2.2]*

Proof. Suppose (M, γ) is a product $R \times I$. View D as $I \times I$ where $I \times \partial I$ is a pair of proper arcs in $R \times \partial I$ and $\partial I \times I$ is a pair of proper arcs in $\partial R \times I$. Since (M', γ') is unchanged under an isotopy of D relative to the sutures, we isotope D to be of the form $\alpha \times I$ where α is a proper arc in R. Cutting along $D = \alpha \times I$ gives $M' = (R \setminus \mathring{N}(\alpha)) \times I$.

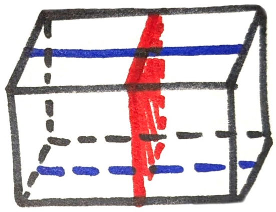


Figure 12: Product disc.

Perhaps a few more words are needed to explain
why gluing back together does not give a twisted product.

Now suppose (M', γ') is a product $R' \times I$. Since M is recovered from M' by gluing back in a thickening of D, it follows that (M, γ) is the product $R \times I$ where R is constructed from R' by attaching a band. $\square$

Theorem 4.4. *A sutured manifold (M, γ) is a product if and only if there is a sequence of product decompositions that terminates in a collection of balls each with a single suture. [Gab86, theorem 1.9]*

Need to worry about the possibility that \gamma is empty

"backwards"

Proof. The *if* direction follows immediately from Lemma 4.3 since the sequence of product decompositions terminates in $E \times I$ where E is a union of discs. Suppose (M, γ) is a product $R \times I$. Choose a family α_i of (pairwise disjoint) proper arcs in R that cut it into a union of discs. Then a sequence of product decompositions along discs $D_i = \alpha_i \times I$ results in a union of copies of $D^2 \times I$. $\square$

The existence of a product disc in (M, γ) tells you that the manifold looks like a product in a neighbourhood of the disc. If the manifold can be decomposed into thickened product discs then it is a product globally.

Coming back to Seifert surfaces, a regular neighbourhood $N(F)$ of a Seifert surface F is homeomorphic to $F \times I$. So the boundary of $S^3 \setminus \mathring{N}(F)$ can be identified with the boundary of $F \times I$. In other words, the exterior of F (the decomposition of S^3 along F) is the sutured manifold *complementary* to $F \times I$.

good

Question: Please give more detail for the idea "induced by going once around the base."

only defined terms/words needs to be in italics

Definition 4.5. *A link L is fibred if the link complement $S^3 \setminus L$ is a fibre bundle over S^1 and the closure of each fibre is a Seifert surface of L. A Seifert surface which occurs in this way is called a fibre surface for L. The monodromy of a fibre is the homeomorphism induced by going once around the base S^1.*

a bit vague

From Example 4.2 we know that the complement of the unknot fibres trivially over the circle. The general method for checking if L is fibred with fibre surface F is to check that the exterior of F is homeomorphic to $F \times I$ [Gab86, theorem 1.9]. It is enough to know that the exterior is a product, which will in practice involve Theorem 4.4, because we already know that the boundary of the exterior is homeomorphic to two copies of F glued along L.

4.2 Examples

To check if a Seifert surface is a fibre:

good. But even better to call this a "procedure", and to give it a name/number so it can be referenced.

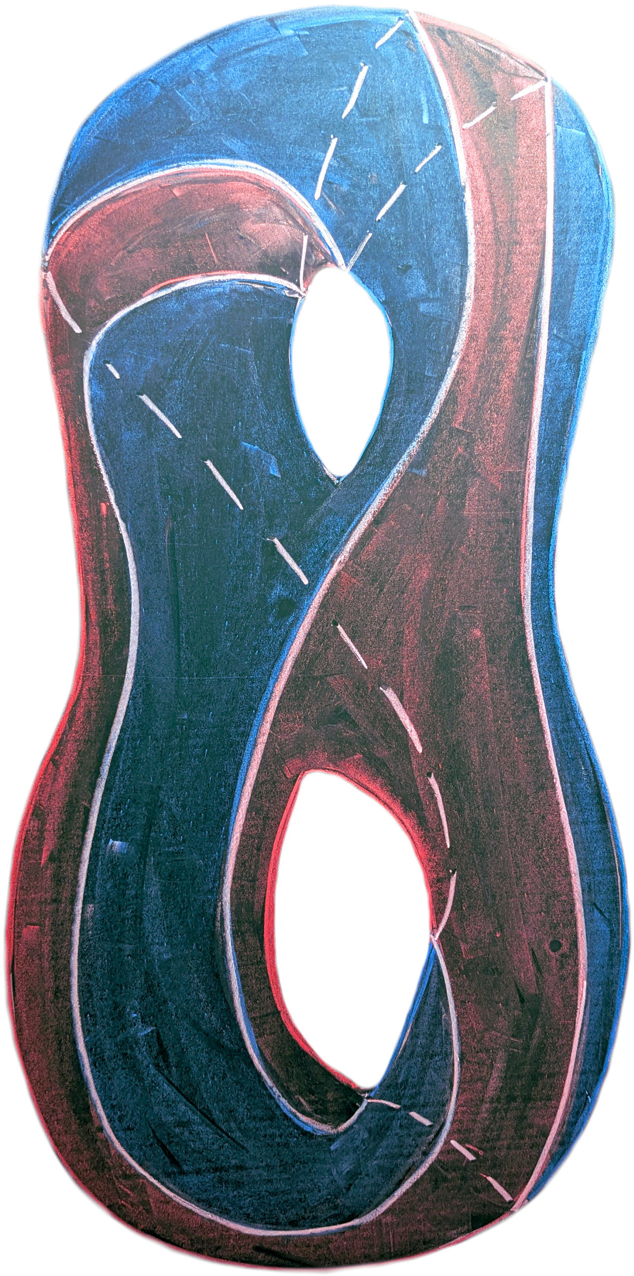
1. Thicken into a sutured product;
2. Take the complement in S^3;
3. If the complement is also a product, then the surface is a fibre.

The Seifert surface in Figure 6 for the left trefoil thickens to the sutured manifold in Figure 14. The obvious product discs fill in the two holes and the result is a ball with a single suture. So the left trefoil is fibred. Similarly, the right trefoil is fibred.

The Seifert surface for the figure-eight in Figure 1 thickens to the sutured manifold in Figure 13. Again, the obvious product discs fill in the two holes and the result is a ball with a single suture. So the figure-eight is fibred. To see the application of product decompositions on another Seifert surface for the figure-eight see Figure 24.

To see *how* the complement of a fibred link is fibred, we can look at how curves flow from one side of the fibre surface to the other under action of going around S^1. For example, in Figure 14 the meridian of the middle handle can be moved to lie on either the positive or negative side. In the next section, the monodromies of the trefoil and figure-eight knots are computed by looking at how arcs (with fixed endpoints) flow through the complement.

All correct, but a bit quick for the very first example! Hmm. I suppose that you do this sort of thing more slowly in Figure 24... So nevermind.



good

Figure 13: Sutured product for figure-eight.

very nice

Figure 14: Sutured product for left trefoil and a loop in the complement.

Question: Give a formula for the inverse of a Dehn twist.

again, this is a bit vague.

5 Genus One Fibred Knots

A fibred link with fibre surface F has a monodromy homeomorphism $h\colon F \to F$ that is defined by going around the base S^1. This homeomorphism fixes the boundary ∂F and is orientation-preserving. As the fibre surface F turns around the link in S^3, any arc α with its endpoints fixed on ∂F is dragged from one side of F to the other. It takes a new position when it returns to F and this new position is exactly the image of the arc under the monodromy $h(\alpha)$. To "see" a piece of the flow we can repeatedly send an arc around and let it drag a surface behind.

At the end of this section, we will see how to compute the monodromy of the trefoil and figure-eight knots as products of *Dehn twists*.

5.1 Dehn Twists

A Dehn twist is a homeomorphism of a surface that is defined by removing an annulus and replacing it with a twist. For an oriented simple closed curve α on a surface F, the Dehn twist τ about α is defined as follows. A regular neighbourhood $N(\alpha)$ is an oriented annulus $S^1 \times I$. Define $\tau\colon F \to F$ to be the identity outside $N(\alpha)$ and given by $(s,t) \mapsto (se^{2\pi i t}, t)$ inside $N(\alpha)$. [Lic97, p. 12.3] The inverse of a Dehn twist is isotopic to the Dehn twist about the same curve but with the twist in the opposite direction. A bit vague... sort of...

There are two standard Dehn twists for the torus. The Dehn twist about the meridian and the Dehn twist about the longitude. We will see that these generate all the orientation-preserving homeomorphisms of the torus, up to isotopy. sentence fragment.

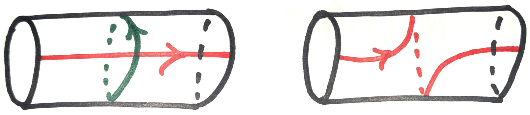


Figure 15: The action of a Dehn twist on a proper arc.

By composing Dehn twists we can build more interesting homeomorphisms of surfaces. The simplicity of the action of a Dehn twist allows us to build up a homeomorphism that has been specified by its action on curves or proper arcs.

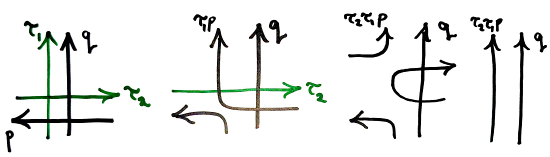


Figure 16: Eliminating an intersection by Dehn twists. [Lic97, p. 12.5]

"Following"

Question: Explain the phrase "turn the torus bag insided out"

Question; You claim that \tau_m \tau_l is a twist about the boundary?

Say we have simple closed curves p and q on F and we want a homeomorphism $h\colon F \to F$ taking the isotopy class of p to the isotopy class of q. By Dehn twists about offset copies of the same curves, we can eliminate an intersection point of p and q.

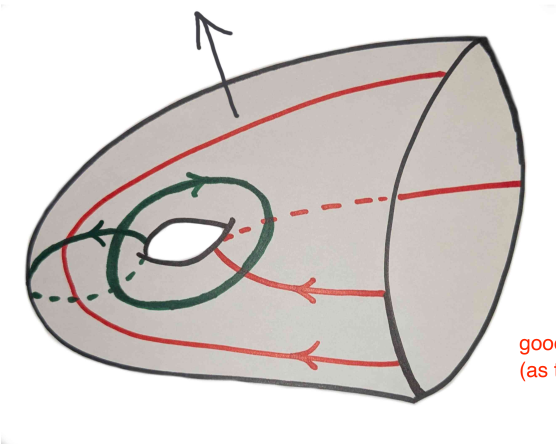
spelling

If p and q intersect tranversely and at precisely one point, then the homeomorphism $\tau_2\tau_1$ (where τ_i are as in Figure 16) does what we want.

This is exactly the situation for the trefoil fibre surface depicted in Figure 14. The meridian loop can be moved in the complement to either the positive or negative side. This gives two closed curves p and q on the fibre surface that intersect in precisely one point. The monodromy takes the curve on the postive side to the curve on the negative side, so one might wonder if it is isotopic to the product $\tau_2\tau_1$. We will see that this is indeed the case.

The genus one surface with one boundary component will be called a *perforated torus* because it is the torus with an open disc removed (see Figure 17). The perforated torus has a meridian and longitude coming from those of the torus, however these can now be interchanged by an isotopy that turns the "torus bag" inside out. So, to be careful about which is the meridian and which is the longitude, we will fix the normal orientation on the perforated torus.

"once-holed torus" is the more standard name



good - better to use small black dots for intersections (as this deals with certain "over/under" problems).

Figure 17: Meridian, longitude and typical proper arcs on the perforated torus.

The composition of the meridian and longitude Dehn twists on the perforated torus is a Dehn twist about a curve parallel with the boundary. No.

Any sufficiently smooth homeomorphism of the torus that fixes a basepoint $0 \in T^2$ is isotopic to a homeomorphism that fixes every point of $N(\{0\}) = D^2$. Such a homeomorphism of the torus induces a homeomorphism of the perforated torus that fixes points on the boundary.

We will now show that the standard Dehn twists generate all the orientation-preserving homeomorphisms of the torus, up to isotopy.

5.2 Mapping Class Group of the Torus

The group $\mathrm{Homeo}(T^2)$ of homeomorphisms of the torus is very large, but we will see that it becomes manageable when we work up to isotopy.

Following [Rol76, p. 2.D], a homeomorphism $h\colon T^2 \to T^2$ induces an automorphism h_* of $\pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$. An automorphism of $\pi_1(T^2)$ can be represented by a matrix that acts on the right. For example, $\begin{pmatrix} 1 & 0 \end{pmatrix}$ is the homotopy class of the meridian of the torus, and

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

This is unfortunate notation...

is the homotopy class for a loop that goes once around the meridian and once around the longitude. So we have a homomorphism $\boxed{*\colon} \mathrm{Homeo}(T^2) \to \mathrm{GL}_2(\mathbb{Z})$ where $\mathrm{GL}_2(\mathbb{Z})$ acts from the right on $\pi_1(T^2)$. The group $\mathrm{GL}_2(\mathbb{Z})$ consists of all integral matrices with determinant ± 1 and is generated by the matrices

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

since these matrices and their inverses generate all row and column operations. good

The images of the meridian and longitude Dehn twists under $*$ are exactly the automorphisms represented by the first two matrices. Also, last matrix represents the image of the inversion map $(x, y) \mapsto (y, x)$ under $*$. Thus the homomorphism $*$ is surjective.

spelling

Lemma 5.1. *The kernal of $*$ is the subgroup of homeomorphisms which are isotopic to the identity. [Rol76, p. 2.D3]*

For the proof we will need Alexander's lemma.

Lemma 5.2. *Every homeomorphism of D^2 that is the identity on ∂D^2 is isotopic (relative to ∂D^2) to the identity. [Rol76, p. 2.D6]*

Proof. Let $f\colon D^2 \to D^2$ be a homeomorphism that restricts to the identity on ∂D. Then

$$f_t(x) = \begin{cases} (1-t) f(x/(1-t)) & 0 \le |x| < 1-t \\ x & 1-t \le |x| \le 1 \end{cases}$$

defines an isotopy between $f = f_0$ and the identity $\mathrm{id}_{D^2} = f_1$. $\square$

Question: Justify the claim of the second paragraph

Proof of Lemma 5.1. [Rol76, p. 2.D5] A homeomorphism of T^2 that is isotopic to the identity is in particular homotopic to the identity and therefore induces the identity automorphism of $\pi_1(T^2)$. Conversely, given a homeomorphism $h\colon T^2 \to T^2$ that induces the identity automorphism on $\pi_1(T^2)$ we will successively *improve* h by isotopy, making it the identity on larger subsets of T^2.

Let M and L be the meridian and longitude of the torus respectively. Since the induced automophism of $\pi_1(T^2)$ is the identity matrix, the image $h(p)$ of any curve p has the same homotopy class as p. So we can begin by isotoping h so that it is the identity on M.

This requires some work...

Next, modify h so that it *also* has $h(N(M)) \subseteq N(M)$. Now we can *straighten out* the annulus $h(N(M))$ so that h is the identity on $N(M)$. Moreover, since any arc embedded in the plane can be straightened to a straight line segment, we can straighten out $h(N(L))$ so that h is now the identity on $N(M) \cup N(L)$.

need to say why h preserves the orientation, sides...

We now have a homeomorphism of T^2 that is the identity on the punctured torus. The closure of the complement of $N(M) \cup N(L)$ is a disc D^2 and h is the identity on ∂D^2. Hence h is isotopic to the identity on T^2. □

The *mapping class group* of a surface S is the group $\mathrm{Mod}(S)$ of orientation-preserving homeomorphism of S that fix the boundary, up to isotopy relative to the boundary.

For example, the mapping class group of an annulus is isomorphic to $\mathbb{Z}$ and is generated by a Dehn twist about curve in the interior that winds around the annulus exactly once.

Theorem 5.3. *The homomorphism $*$ gives an isomorphism from $\mathrm{Mod}(T^2)$ to $\mathrm{SL}_2(\mathbb{Z})$ and $\mathrm{Mod}(T^2)$ is generated by the meridian and longitude Dehn twists.*

Proof. By Lemma 5.1, the homomorphism $*$ gives an isomorphism

$$\mathrm{Homeo}(T^2)/\ker * \to \mathrm{GL}_2(\mathbb{Z})$$

where the quotient is exactly the group of isotopy classes of homeomorphisms of T^2. The orientation-preserving homeomorphisms correspond to the subgroup $\mathrm{SL}_2(\mathbb{Z})$ which is generated by the matrices

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

move up

□

The inverse of this isomorphism can be described as follows. A matrix $A \in \mathrm{SL}_2(\mathbb{Z})$, thought of as a homeomorphism of $\mathbb{R}^2$ that fixes the origin, descends to a homeomorphism h_A of the torus that fixes the basepoint $0 \in T^2$. This is a homomorphism from $\mathrm{SL}_2(\mathbb{Z})$ to $\mathrm{Homeo}(T^2)$. [CB88] For example, the generators of $\mathrm{SL}_2(\mathbb{Z})$ above descend to homeomorphisms that are

true but not proven
(and not true in higher
genus...)

isotopic to the meridian and longitude Dehn twists respectively. Theorem 5.3 tells us that $(h_A)_* = A$ because it is true for the generators.

Since every homeomorphism of T^2 is isotopic to a homeomorphism that fixes $0 \in T^2$, the isotopy classes of homeomorphisms of the punctured torus $T^2 \setminus \{0\}$ are in bijection with the isotopy classes of homeomorphisms of T^2. So the mapping class group of $T^2 \setminus \{0\}$ is also $\mathrm{SL}_2(\mathbb{Z})$ and is generated by the meridian and longitude Dehn twists. This is seen as a consequence of a Dehn twist about a curve that is isotopic to the puncture being trivial in the mapping class group.

ok - depends on conventions

However, removing a disc is not the same as removing a point because a Dehn twist about a curve that is isotopic to the boundary is not trivial. So a homeomorphism of the perforated torus does not induce a unique homeomorphism of the torus. Instead it is only unique up to conjugation (the conjugation is akin to a change of basis that moves the boundary away and then back).

no

5.3 Classifying the Homeomorphisms of the Torus

There are three cases for a matrix $A \in \mathrm{SL}_2(\mathbb{Z})$:

1. Complex nonreal eigenvalues,
2. Repeated eigenvalue,
3. Distinct real eigenvalues.

Since $\det(A) = 1$, the characteristic polynomial of A is $\lambda^2 - \mathrm{tr}(A)\lambda + 1$. So the three cases correspond to $|\mathrm{tr}(A)| < 2$, $|\mathrm{tr}(A)| = 2$ and $|\mathrm{tr}(A)| > 2$.

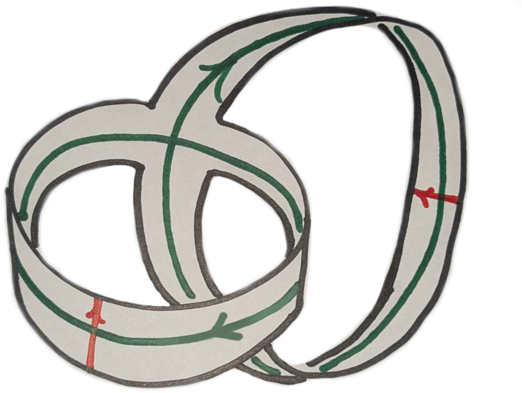
In the first case, there are only two possibilities for $\mathrm{tr}(A)$. Either $\mathrm{tr}(A) = 0$ and the eigenvalues are $\pm i$, or $\mathrm{tr}(A) = \pm 1$. If $\mathrm{tr}(A) = 0$ then Cayley-Hamilton theorem gives $A^2 = -I$. Similarly, if $\mathrm{tr}(A) = -1$ then $A^3 = I$, and if $\mathrm{tr}(A) = 1$ then $A^3 = -I$. So in this case the homeomorphism h_A is periodic.

In the second case, we have $\mathrm{tr}(A) = \pm 2$. Since the trace is the sum of the eigenvalues, the repeated eigenvalue is ± 1. In this case the homeomorphism $h_A \colon T^2 \to T^2$ is said to be *reducible*. [CB88, page 2].

In the third case, the product of the eigenvalues is $\det(A) = 1$, so there is a contracting eigenvalue $|\lambda_1| < 1$ and an expanding eigenvalue $|\lambda_2| > 1$. Every translate of the contracting (resp. expanding) eigenspace is invariant under h_A and is contracted (resp. expanded) by h_A. In this case the homeomorphism h_A is said to be *Anosov*.

5.4 Action on Proper Arcs

We want to be able to identify a homeomorphism by how it acts on proper arcs.



good

Figure 18: Closed curves and proper arcs on a perforated torus.

The arcs in Figure 18 cut the surface into a disc. Suppose h is an orientation-preserving homeomorphism of the surface. Then the images of these arc under h are also arcs, and they are cut the surface into a disc. So if we know where these arcs go under the homeomorphism h then we get a homeomorphism of the disc that fixes its boundary. So Alexander's lemma tells us that there is a unique homeomorphism that acts on the arcs in the specified way. [FM12, p. 2.3]

5.5 Examples

The matrix representing a homeomorphism of the torus is only uniquely defined when a basis for $\pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$ is specified. In the following computations, the choice of meridian and longitude Dehn twist has been decided by what is convienient for the Seifert surface.

The left and right trefoil have similar monodromy (simply swap all the crossings in the following computation) so only the computation for the right trefoil is given.

5.5.1 Trefoil

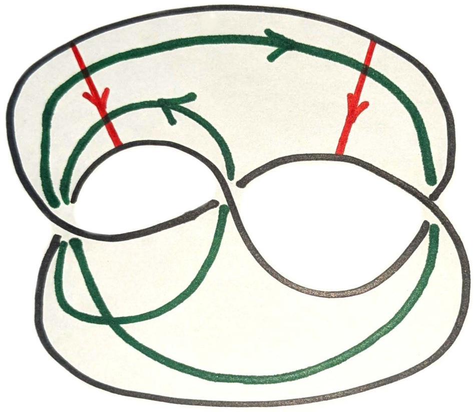
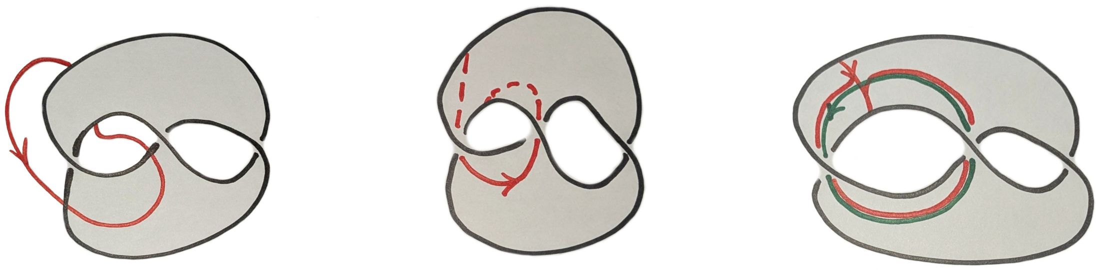


Figure 19: Fibre surface of the right trefoil.

Using product decompositions we can quickly determine that the surface in Figure 19 is the fibre surface for the right trefoil. To know the monodromy it is enough to know where the arcs go when they are lifted off the surface and placed back on the other side. Figures 20a and 20b show the first arc being lifted up and back on the other side. The new arc is seen to be the result of a Dehn twist in Figure 20c.



(a) (b) (c)

Figure 20: Image of the first arc under the monodromy.

Similarly, the image of the second arc under the monodromy is given by the Dehn twist in Figure 21c. Here we see that it is actually the inverse of the standard Dehn twist depicted in Figure 19.

(a) (b) (c)

Figure 21: Image of the second arc under the monodromy.

Since the image of the first arc is unchanged by the second Dehn twist, as shown in Figure 22, it follows that the monodromy of the right trefoil is exactly the product of these Dehn twists.

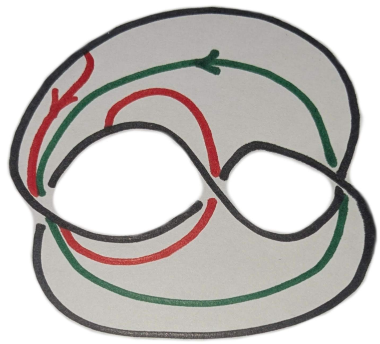


Figure 22: The arc does not intersect with the curve.

The monodromy of the right trefoil is therefore given by the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

This is periodic.

5.5.2 Figure-eight

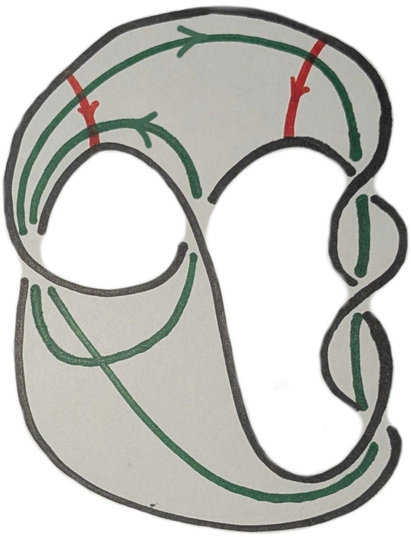
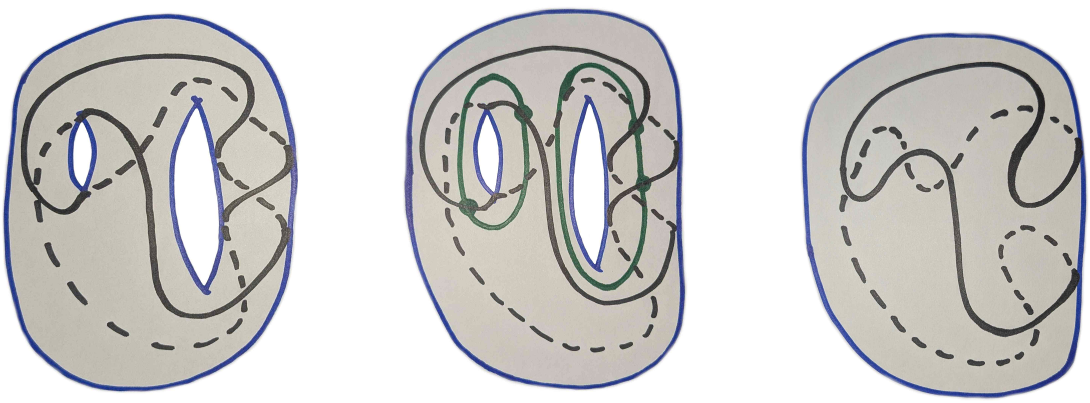


Figure 23: Fibre surface of the figure-eight.

The fibre surface in Figure 23 does not look like the fibre surface we have already seen. The product decompositions in Figure 24 show that it is indeed a fibre surface and therefore *the* fibre surface.



(a) (b) (c)

Good

Figure 24: Product decompositions to check that the surface is a fibre.

The image of the first arc under the monodromy is as it was for the trefoil so is given by the first Dehn twist in Figure 23. It is harder to see where the second arc goes. The first step is shown in Figure 25.

Question: Explain why the monodromy is orientation preserving (give a proof directly from the discussion about arcs).

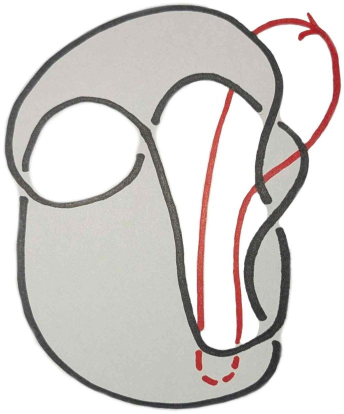
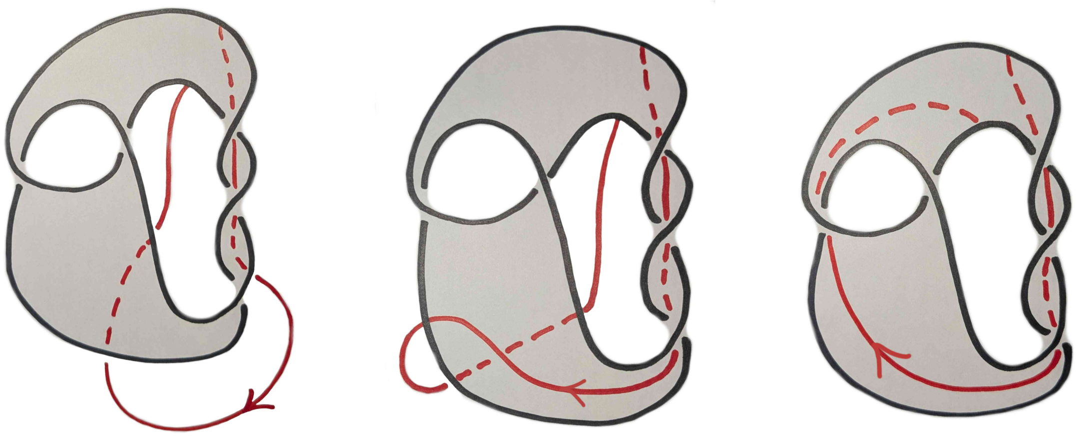


Figure 25: Lifting the second arc off the surface.

The remaining steps are shown in Figure 26. The result is the same as the action of the second Dehn twist in Figure 23. The only difference from the case of the trefoil is the direction of the Dehn twist. For the trefoil it was the inverse of this standard Dehn twist.



very nice

(a) (b) (c)

Figure 26: Image of the second arc under the monodromy.

Again, since the image of the first arc is unchanged by the second Dehn twist, it follows that the monodromy of the figure-eight is exactly the product of these Dehn twists. The monodromy of the figure-eight is therefore given by the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

This is Anosov.

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