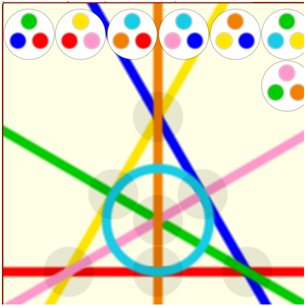
From Dobble to Klein

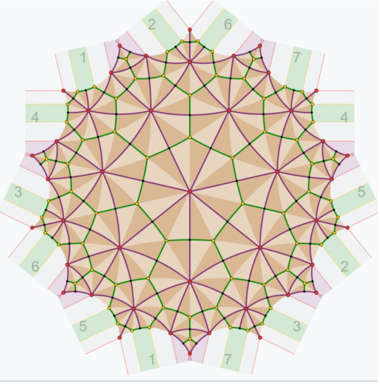


Have a go with a Dobble/Fano inspired game
There is also a “make your own Mini Dobble” handout

Starting points

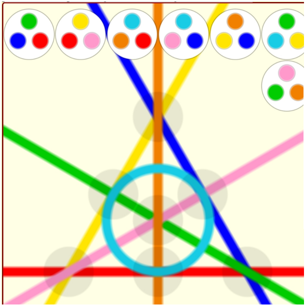
What do the following have in common?

Klein's quartic



(from Wikipedia)

Fano's plane



(from my web page)

Answer

- Question: What do the Klein Quartic and the Fano plane have in common?

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Answer

- Question: What do the Klein Quartic and the Fano plane have in common?
- $$PSL(7,2) \cong PGL(3,2)$$
- They have the same automorphism groups!
- And these are they!
- But what are these objects, and what are automorphims groups?

Dobble and the Fano plane (see also MA243)

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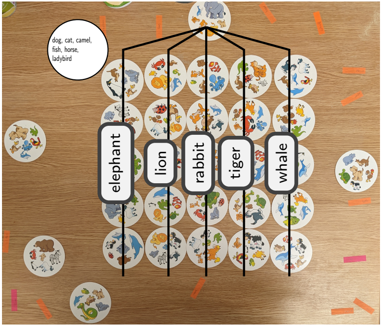

lines of the form $\{(\alpha s : t : s) : (t : s) \in \mathbb{P}^1\}$

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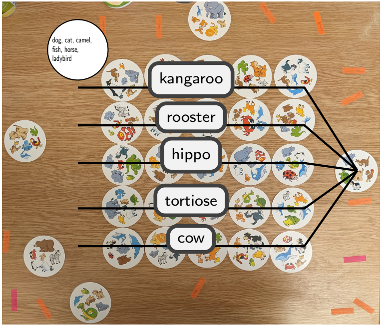

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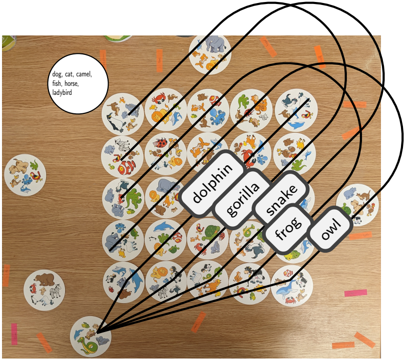

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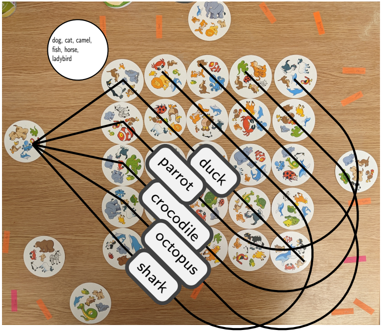

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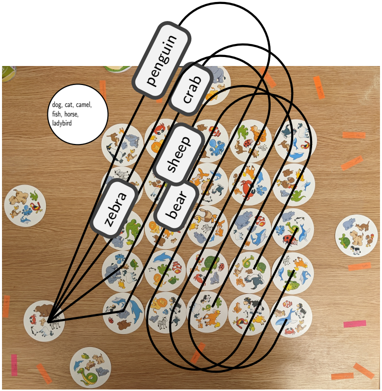

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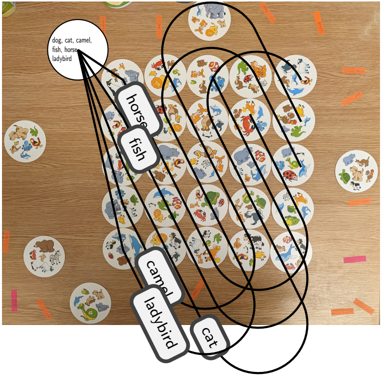

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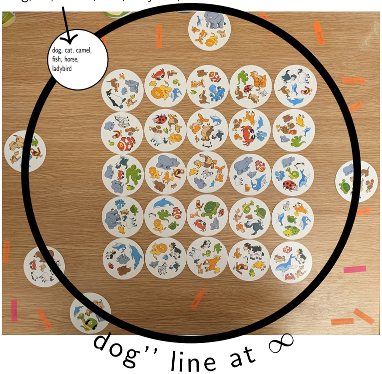
Dobble and the Fano plane (see also MA243)

Dobble works because every pair of cards have a common symbol

- There are 6 sets of "parallel" lines
- And a line at "infinity"

missing card:

dog,cat, camel, fish, ladybird, horse



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$(0:1:0)$

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	(0,1)	(1,1)	(2,1)	(3,1)	(4,1)	
$(1:2:0)$	(0,0)	(1,0)	(2,0)	(3,0)	(4,0)	

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interpret (x, y) as $(x : y : 1)$; all computations mod 5

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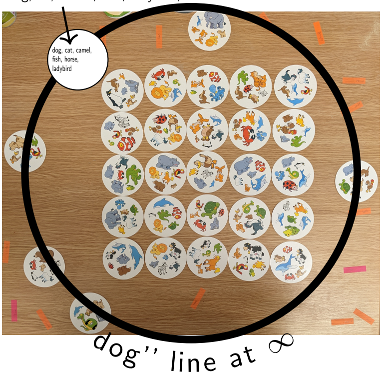

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Note that these cards could have been laid out in 372000 different ways, even with the cards all in the same grid pattern (this statement needs explaining).

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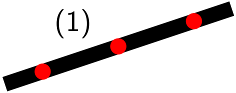
- (1) Every line contains at least three points

- 

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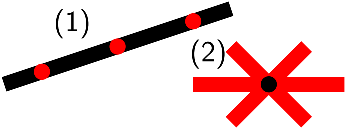

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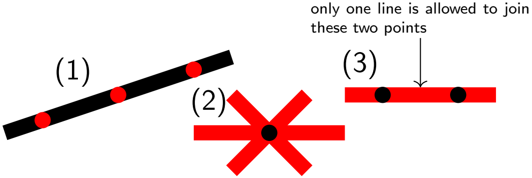
- 

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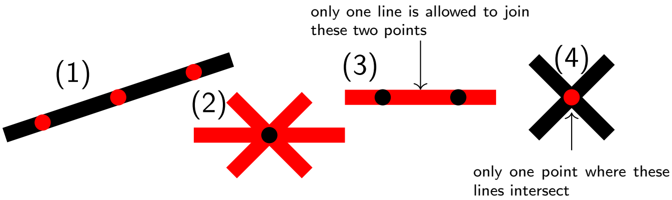

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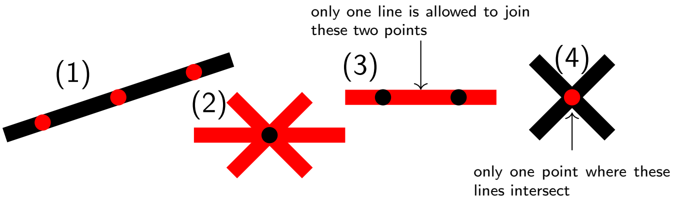
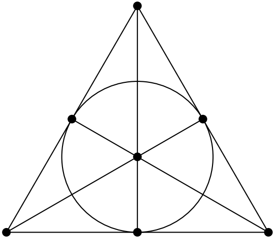


- If these axioms hold, for some n, there are $n+1$ points on each line, $n+1$ lines through each point, and n^2+n+1 lines and n^2+n+1 points. n is the **order** of the plane.

The Fano Plane

The smallest axiomatic projective plane is the **Fano Plane** with order 2:



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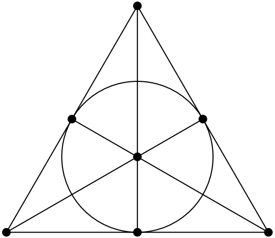

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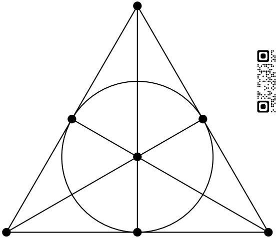
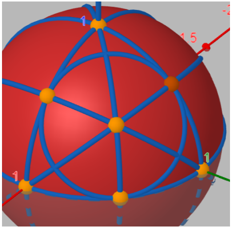
https://www.geogebra.org/3d/xfbucsjk





- What's projective about this?
- The vectors $(1:0:0), (0:1:0), (0:0:1), (1:1:0), (1:0:1), (0:1:1), (1:1:1)$ project to these points on a sphere. (... projective geometry...)

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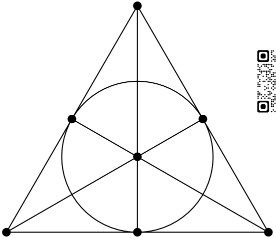

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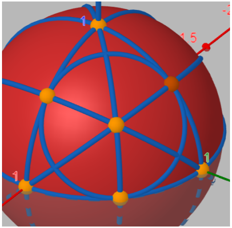

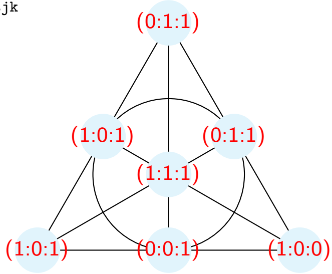

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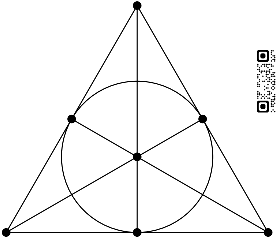

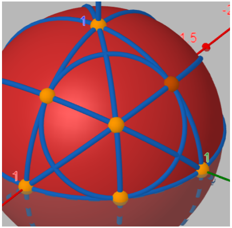

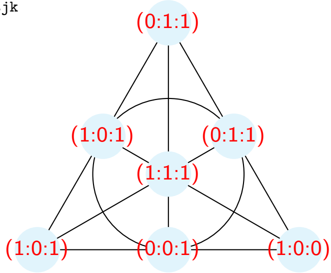

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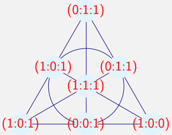
- You can use these coordinates to describe these points.
- We only need to think of these modulo 2 for the Fano Plane

Automorphisms of the Fano Plane



Let's call the Fano plane

$$\mathbb{P}^2(\mathbb{F}_2)$$

for short (this is actually only the set of points; the Fano plane is a set of the form (points, lines).)

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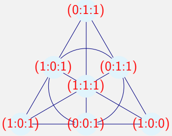

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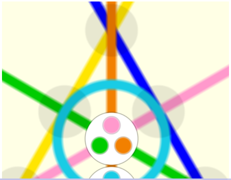

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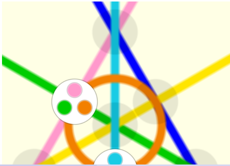

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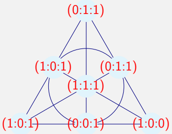

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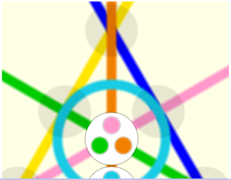

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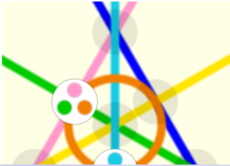

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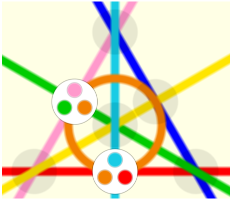

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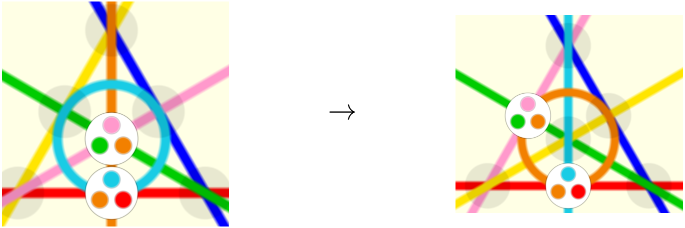

- For the Fano plane, such maps can be described in terms of 3×3 invertible matrices modulo 2, with entries being 0 or 1.
- Corresponds to choosing the position of three "independent" points
- The choices correspond to different possible basis of $\mathbb{F}_2^3$

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$$PGL(3, 2)$$

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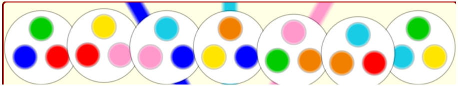
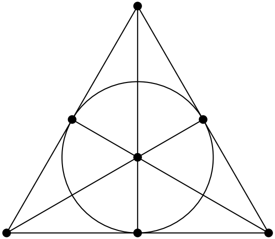
- This group has order 168
- So there are 168 ways to put these points



on this diagram (in such a way that...)



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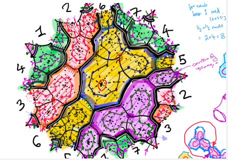
- With an edge between two elements if they are related by a generator...

$PSL(2,7)$ and the Klein Quartic



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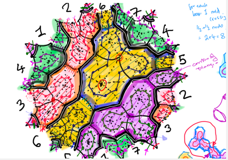

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$$x^3y + y^3z + z^3x = 0$$

(projective, complex coords)

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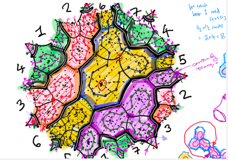

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  https://en.wikipedia.org/wiki/Klein_quartic

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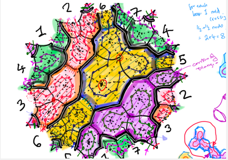

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- This is isometric to a certain quotient of the Poincare disc...

  `https://en.wikipedia.org/wiki/Klein_quartic`
- The automorphism group of the Klein quartic is relatively easy to understand from the fundamental domain in the Poincare disc, because each element corresponds to one triangle.